

# A SPECTRAL DECOMPOSITION FOR SELF-ADJOINT ELEMENTS IN THE MAXIMUM GCR IDEAL OF A VON NEUMANN ALGEBRA WITH APPLICATIONS TO NONCOMMUTATIVE INTEGRATION THEORY<sup>(1)</sup>

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**1. Introduction.** Let  $\mathcal{A}$  be a von Neumann algebra of Type I with center  $\mathcal{Z}$  and let  $I_a$  be the ideal in  $\mathcal{A}$  generated by the abelian projections of  $\mathcal{A}$ . The ideal  $I_a$  is the natural analogue in a von Neumann algebra of the ideal of completely continuous operators on a Hilbert space. Here, however, the center  $\mathcal{Z}$  functions in the same way that the complex numbers function in the case of completely continuous operators. In fact, we prove that every positive operator  $A$  in  $I_a$  may be written in the form  $A = \sum \{A_j E_j \mid j=1, 2, \dots\}$  where  $\{E_j \mid j=1, 2, \dots\}$  is a sequence of mutually orthogonal abelian projections such that  $E_1 > E_2 > \dots$  and  $\{A_j \mid j=1, 2, \dots\}$  is a sequence of positive central elements such that  $A_1 \geq A_2 \geq \dots$  and  $\lim_j A_j = 0$ . This decomposition for  $A$  will be unique in the same sense that the spectral decomposition for completely continuous operators is unique.

The ideal  $I_a$  is the maximum GCR ideal in the von Neumann algebra  $\mathcal{A}$ . In fact,  $I_a$  is a CCR ideal and it has a Hausdorff structure space. In §3 we show that the structure space of  $I_a$  can be identified with the spectrum of the center  $\mathcal{Z}$ . Then we describe the positive elements  $A$  in  $I_a$  and in  $\mathcal{A}$  with continuous trace i.e. those positive elements  $A$  such that  $\phi \rightarrow \text{Trace}(\phi(A))$  is a continuous function from the space of equivalence classes of irreducible representations of  $I_a$  and  $\mathcal{A}$  respectively to the complex field.

In §4 we define in  $\mathcal{A}$  analogues  $\mathcal{S}$  and  $\mathcal{T}$  to the Schmidt class and trace class in the algebra of all bounded linear operators on a Hilbert space. The sets  $\mathcal{S}$  and  $\mathcal{T}$  are ideals of  $\mathcal{A}$  contained in  $I_a$  and  $\mathcal{S}^2 = \mathcal{T}$ . Furthermore, a trace  $\text{Tr}$  (with values in  $\mathcal{Z}$ ) is defined on  $\mathcal{T}$ . This trace can be extended to a normal trace on  $\mathcal{A}^+$ . The ideal  $\mathcal{S}$  under an inner product defined by  $\text{Tr}$  becomes a  $\mathcal{Z}$ -module [9] and the set of all uniformly continuous functions on  $I_a$  into  $\mathcal{Z}$  that are linear over  $\mathcal{Z}$  is identified with  $\mathcal{T}$ . Also, a norm equal to the operator bound on  $\mathcal{T}$  gives a Banach  $*$ -algebra structure on  $\mathcal{T}$ . With this norm the set of all continuous functions of  $\mathcal{T}$  into  $\mathcal{Z}$  that are linear over  $\mathcal{Z}$  is identified with  $\mathcal{A}$  and the operator bound is the norm defined on  $\mathcal{A}$ . The positive elements in  $\mathcal{T}$  are identified with the normal function of  $\mathcal{A}$  into  $\mathcal{Z}$  which are linear over  $\mathcal{Z}$ . These are analogues of results

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concerning scalar valued traces on von Neumann algebras and results on completely continuous operators on Hilbert space.

Finally, we justify the use of the name Schmidt class and trace class by showing that these ideals contain all other ideals which produce  $\mathcal{L}$ -modules. We also show the way in which this theory differs from the usual noncommutative integration theory.

**2. Diagonalization of self-adjoint operators in  $I_a$ .** Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $Z$  be the spectrum of  $\mathcal{Z}$ . For each  $\zeta \in Z$  define  $[\zeta]$  to be the closed two-sided ideal given by

$$[\zeta] = \text{closure } \{ \sum \{ A_j B_j \mid 1 \leq j \leq n \} \mid A_j \in \mathcal{A}, B_j \in \zeta, n \text{ a positive integer} \}.$$

There is for each  $\zeta \in Z$  an irreducible representation  $\psi_\zeta$  of  $\mathcal{A}$  whose kernel is  $[\zeta]$  on the Hilbert space  $H(\zeta)$ . We denote the image of  $A$  in  $\mathcal{A}$  under  $\psi_\zeta$  by  $A(\zeta)$ . Then the function  $\zeta \rightarrow \|A(\zeta)\|$  of  $Z$  into the positive real numbers is a continuous function [4, §4]. The image of  $I_a$  under  $\psi_\zeta$  is the ideal of all completely continuous operators of  $H(\zeta)$ . If  $\mathcal{A}$  is homogeneous of finite degree  $n$ , the Hilbert space  $H(\zeta)$  is  $n$ -dimensional.

We shall use the following terminology. Let  $\mathcal{Z}$  be a commutative von Neumann algebra with spectrum  $Z$ . The projection in  $\mathcal{Z}$  whose image under the Gelfand isomorphism  $A \rightarrow A^\wedge$  of  $\mathcal{Z}$  onto the set of continuous complex-valued functions on  $Z$  is the characteristic function of an open and closed set in  $Z$  will be called the projection corresponding to  $W$ .

In order to obtain the decomposition for self-adjoint elements in  $I_a$  we need the following lemma.

**LEMMA.** *Let  $\mathcal{A}$  be a finite Type I von Neumann algebra which has only finitely many homogeneous components of distinct degrees. Let  $\mathcal{Z}$  be the center of  $\mathcal{A}$  and let  $Z$  be the spectrum of  $\mathcal{Z}$ . Let  $A$  be a self-adjoint element of  $\mathcal{A}$  and let  $A_1$  be an element in  $\mathcal{Z}$  such that for each  $\zeta$  in  $Z$  the number  $A_1^\wedge(\zeta)$  is a proper value of the matrix  $A(\zeta)$  on the finite-dimensional Hilbert space  $H(\zeta)$ . Then there is an abelian projection  $E$  in  $\mathcal{A}$  of central support 1 such that  $AE = A_1 E$  [1].*

The following theorem allows us to apply the results of this lemma.

**THEOREM 2.1.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra on the Hilbert space  $H$ ; a projection  $E$  in the ideal  $I_a$  generated by the abelian projections of  $\mathcal{A}$  is the sum of a finite number of mutually orthogonal abelian projections.*

**Proof.** The algebra  $E\mathcal{A}E$  is a Type I von Neumann algebra on  $E(H)$ . If  $\phi$  is an irreducible representation of  $E\mathcal{A}E$  on a Hilbert space  $K$ , then  $K$  is finite dimensional. Indeed, there is an irreducible representation  $\phi'$  of  $\mathcal{A}$  on a Hilbert space  $K'$  which contains  $K$  as a subspace such that  $\phi'(E)$  is the projection corresponding to  $K$  and  $\phi'(A)x = \phi(A)x$  for every  $x \in K$  and  $A \in E\mathcal{A}E$ . We have that  $\phi'(I_a)$  is the set of

completely continuous operators on  $K'$ . This means that  $\phi'(E)$  is a finite-dimensional projection and, therefore, that  $K$  is finite dimensional.

Let  $\{P_n\}$  be a net of mutually orthogonal projections in the center of  $E\mathcal{A}E$  such that the least upper bound of the  $P_n$  is  $E$  and  $E\mathcal{A}EP_n$  is homogeneous. Each  $E\mathcal{A}EP_n$  must have finite degree and so we may assume  $\{P_n\}$  is either a finite set or an infinite sequence such that if  $f(n)$  is the degree of  $E\mathcal{A}EP_n$  then  $f(n) < f(n+1)$  ( $n=1, 2, \dots$ ). It is sufficient to prove that the set  $\{P_n\}$  is a finite set. We argue by contradiction. If there are an infinite number of  $\{P_n\}$  there is a point  $\zeta$  in the spectrum of the center of  $E\mathcal{A}E$  such that  $P_n^\wedge(\zeta)=0$  for all  $n$ . Let  $\{E_{nj} \mid 1 \leq j \leq f(n)\}$  be mutually orthogonal equivalent abelian projections of sum  $P_n$ . Let  $F_j = \sum \{E_{nj} \mid k+1 \leq n\}$  whenever  $f(k)+1 \leq j \leq f(k+1)$  for  $k=0, 1, 2, \dots$  where  $f(0)=0$ . Each  $F_j$  is an abelian projection; if  $f(k)+1 \leq j \leq f(k+1)$  the central support of  $F_j$  is  $\sum \{P_n \mid k+1 \leq n\}$ . We have that  $(\sum \{P_n \mid n \leq k\})^\wedge(\zeta)=0$  and so  $(\sum \{P_n \mid k+1 \leq n\})^\wedge(\zeta)=1$ . Thus  $F_j(\zeta) \neq 0$  for  $1 \leq j < \infty$ . There is an irreducible representation with kernel  $[\zeta]$  of  $E\mathcal{A}E$  on a Hilbert space  $H(\zeta)$ . This means that  $\sum \{F_j(\zeta) \mid 1 \leq j \leq f(k)\} \leq E(\zeta)$  for all  $k$ . Because  $\{F_j(\zeta)\}$  is a sequence of nonzero mutually orthogonal projections,  $H(\zeta)$  must be infinite dimensional. This is impossible. So there are only finitely many  $P_n$ .

We now apply Lemma and Theorem 2.1 to the spectral decomposition of a self-adjoint operator in  $I_a$  to obtain a diagonalization. By a rearrangement of this diagonalization we prove a spectral theorem analogous to the spectral theorem for self-adjoint completely continuous operators.

**THEOREM 2.2.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $I_a$  be the ideal generated by the abelian projections of  $\mathcal{A}$ . Let  $A$  be a positive element in  $I_a$ . There is an at most denumerable set  $\{E_n\}$  of mutually orthogonal abelian projections such that  $E_1 > E_2 > \dots$  and at most denumerable set  $\{A_n\}$  of positive elements in  $\mathcal{Z}$  such that  $A_1 \geq A_2 \geq \dots$  and such that  $\lim_n A_n = 0$  if there are infinitely many  $A_n$  with the property  $A = \sum_n A_n E_n^{(2)}$ .*

**Proof.** Let  $Z$  be the spectrum of the center  $\mathcal{Z}$ . Because the function  $\zeta \rightarrow \|A(\zeta)\|$  of  $Z$  into the real numbers is continuous, there is an element  $A_1$  in  $\mathcal{Z}^+$  such that  $A_1^\wedge(\zeta) = \|A(\zeta)\|$  for every  $\zeta \in Z$ . There is no loss of generality in the assumption that  $\|A\| = 1$ . Let  $X'_n = \text{closure } \{\zeta \in Z \mid A_1^\wedge(\zeta) > (n+1)^{-1}\}$  for  $n=1, 2, \dots$ . The sets  $X'_n$  are open and closed subsets of  $Z$ . By induction we may define the sets  $X_1 = X'_1$  and  $X_j = X'_j - \bigcup \{X_k \mid 1 \leq k < j\}$  for  $j=2, 3, \dots$ . The sets  $X_n$  are mutually disjoint and are open and closed. Let  $P_n$  be the central projection which corresponds to  $X_n$  for each  $n=1, 2, \dots$ .

Let  $n$  be a fixed natural number and let  $B = AP_n$ . Let  $\mathcal{B}$  be a maximal commutative \*-subalgebra of  $\mathcal{A}P_n$  that contains  $B$  and let  $Z(\mathcal{B})$  denote the spectrum of  $\mathcal{B}$ . Let  $E$  be the projection in  $\mathcal{B}$  which corresponds to the open and closed set closure  $\{\zeta' \in Z(\mathcal{B}) \mid B^\wedge(\zeta') > (2(n+1))^{-1}\}$ . The spectral projection  $E$  of  $B$  has the property

(2) I wish to thank the referee for suggesting that a proof along the following lines might be possible. I had originally deduced the theorem from the lemma in a more computational manner.

$B \geq (2(n+1))^{-1}E$  and  $B(1-E) \leq (2(n+1))^{-1}(1-E)$ . The central support of  $E$  in  $\mathcal{A}$  is  $P_n$ . Indeed, if  $E(\zeta)=0$  for some  $\zeta \in X_n$ , then  $\|B(\zeta)\| = \|B(1-E)(\zeta)\| \leq (2(n+1))^{-1}$ . This is impossible. Thus  $E(\zeta) \neq 0$  for every  $\zeta \in X_n$ . Now  $B \geq (2(n+1))^{-1}E$  implies that there is an element  $C$  in  $\mathcal{B}$  such that  $CB=E$ . Therefore  $E$  is a projection in  $I_a$ . We have that  $\|B(\zeta)\| = \max \{\|BE(\zeta)\|, \|B(1-E)(\zeta)\|\}$  for every  $\zeta \in X_n$ . Since

$$\|B(1-E)(\zeta)\| \leq (2(n+1))^{-1},$$

we find that  $\|B(\zeta)\| = \|BE(\zeta)\|$  for every  $\zeta \in X_n$ . Let  $B' = BE$ . By Theorem 2.1 the Type I algebra  $\mathcal{A}_E$  is finite with finitely many homogeneous components of distinct degrees. Let  $Z(\mathcal{A}_E)$  be the spectrum of the center of  $\mathcal{A}_E$ . The function  $\zeta'' \rightarrow \|B'(\zeta'')\|$  is continuous on  $Z(\mathcal{A}_E)$  and there is an element  $B_1$  in the center of  $\mathcal{A}_E$  such that  $B_1 \hat{(\zeta'')} = \|B'(\zeta'')\|$  for every  $\zeta'' \in Z(\mathcal{A}_E)$ .

There is an abelian projection  $F = F_n$  in  $\mathcal{A}_E$  of central support  $E$  such that  $BF = B_1F$ . Considering  $F$  as a projection of  $\mathcal{A}$ , we see that  $F$  is an abelian projection of  $\mathcal{A}$  of central support  $P_n$ . The center of  $\mathcal{A}_E$  is  $\mathcal{Z}E$  and the function  $\rho(C) = CE$  of  $\mathcal{Z}P_n$  onto  $\mathcal{Z}E$  is an isomorphism. Let  $A'_1$  be the unique element in  $\mathcal{Z}P_n$  such that  $A'_1E = B_1$ ; we show that  $A'_1 = A_1P_n$ . Let  $\zeta \in X_n$  and let  $\zeta'$  be the maximal ideal in  $\mathcal{Z}P_n$  given by  $\zeta' = \{CP_n \mid C \in \zeta\}$ ; then  $A'_1 \hat{(\zeta)} = B_1 \hat{(\rho(\zeta'))} = \|B'(\rho(\zeta'))\| = \inf \{\|B' + D\| \mid D \in [\rho(\zeta')]\} \geq \inf \{\|B' + D\| \mid D \in [\zeta]\} = \|B(\zeta)\| = A_1 \hat{(\zeta)}$  because the ideal  $[\rho(\zeta')]$  in  $\mathcal{A}_E$  is contained in the ideal  $[\zeta]$  in  $\mathcal{A}$ . So  $A'_1 \geq A_1P_n$ . However, if  $\zeta \in X_n$  and if  $D \in [\zeta]$  it is easy to see that  $EDE \in [\rho(\zeta')]$ . This means that  $A'_1 \hat{(\zeta)} = \|B'(\rho(\zeta'))\| \leq \|EBE - EDE\| \leq \|B - D\|$ , for every  $D \in [\zeta]$ . Thus,  $A'_1 \hat{(\zeta)} \leq \|B(\zeta)\| = A_1 \hat{(\zeta)}$  for every  $\zeta \in X_n$ . So  $A'_1 \leq A_1P_n$ . Consequently, we have found an abelian projection  $F_n$  of central support  $P_n$  such that  $AF_n = A_1F_n$ .

Let  $P = \sum \{P_n \mid n=1, 2, \dots\}$  and let  $E_1 = \sum \{F_n \mid n=1, 2, \dots\}$ ; the projection  $E_1$  is abelian and has central support  $P$  which corresponds to the closure of the set  $\{\zeta \in Z \mid \|A(\zeta)\| \neq 0\}$ . Furthermore,  $AE_1 = A_1E_1$ . Then the operator  $A(1-E_1)$  is a positive operator in  $I_a$  and for every  $\zeta$  in  $Z$  the relation  $\|A(1-E_1)(\zeta)\| \leq \|A(\zeta)\| \cdot \|(1-E_1)(\zeta)\| \leq A_1 \hat{(\zeta)}$  is true. By induction we can construct a sequence  $\{A_n\}$  of positive central elements such that  $A_1 \geq A_2 \geq \dots$  and a sequence of mutually orthogonal abelian projections  $\{E_n \mid n=0, 1, 2, \dots\}$  such that

- (i)  $E_0 = 0$ ;
- (ii)  $AE_n = A_nE_n$  ( $n=1, 2, \dots$ );
- (iii) The function  $\zeta \rightarrow \|A(1 - \sum \{E_j \mid 0 \leq j \leq n-1\})(\zeta)\|$  on  $Z$  is equal to  $A_n \hat{(\zeta)}$  ( $n=1, 2, \dots$ ); and
- (iv) The central support  $Q_n$  of  $E_n$  corresponds to the closure of the set

$$\{\zeta \in Z \mid A_n \hat{(\zeta)} \neq 0\}.$$

In view of (iii) in order to show  $A = \sum_n A_nE_n$  (uniformly), it is sufficient to show that  $\lim_n A_n = 0$  (uniformly). Indeed,

$$\|A - \sum \{A_jE_j \mid 1 \leq j \leq n\}\| = \|A(1 - \sum \{E_j \mid 1 \leq j \leq n\})\| = \|A_{n+1}\|.$$

We prove  $\lim_n A_n = 0$  by arguing by contradiction. Suppose  $\lim_n A_n \neq 0$ . Taking into consideration the ordering of the sequence  $\{A_n\}$ , there is a number  $\delta > 0$  such that  $\|A_n\| > \delta$  for every  $n$ . So for each  $n$  there is a maximal ideal  $\zeta_n$  in  $Z$  such that  $A_n^\wedge(\zeta_n) \geq \delta$ . For  $j = 1, 2, \dots$ ,  $\delta \leq A_{n+j}^\wedge(\zeta_{n+j}) \leq A_n^\wedge(\zeta_{n+j})$ . Let  $\zeta$  in  $Z$  be a limit point of the set  $\{\zeta_n \mid n = 1, 2, \dots\}$ . For every  $j = 1, 2, \dots$  we have that  $A_j^\wedge(\zeta) \geq \liminf_n A_j^\wedge(\zeta_n) \geq \delta$ . Because each  $A_j^\wedge(\zeta)$  is not zero, each  $E_j(\zeta)$  is a one-dimensional projection on the Hilbert space  $H(\zeta)$ . The sequence  $\{E_n(\zeta)\}$  is a sequence of orthogonal one-dimensional projections on  $H(\zeta)$  such that  $A(\zeta)E_j(\zeta) = A_j^\wedge(\zeta)E_j(\zeta)$  for each  $j = 1, 2, \dots$ . This shows that the compact operator  $A(\zeta)$  has infinitely many proper values which are greater than or equal to the strictly positive number  $\delta$ . This is impossible. Hence, we conclude  $\lim_n A_n = 0$ .

Finally, if  $Q_n$  is the central support of  $E_n$  ( $n = 1, 2, \dots$ ), then  $Q_n$  corresponds to the set closure  $\{\zeta \in Z \mid A_n^\wedge(\zeta) \neq 0\}$ . So  $Q_1 \geq Q_2 \geq \dots$ . Thus  $E_1 > E_2 > \dots$ .

Using the spectral theorem for completely continuous self-adjoint operators on a Hilbert space, we prove that the representation obtained in Theorem 2.2 is unique.

**THEOREM 2.3.** *Let  $A$  be a positive element in  $I_a$  and let  $\{A_j \mid 1 \leq j < m\}$  (respectively,  $\{B_j \mid 1 \leq j < n\}$ ) be a set of positive central elements and  $\{E_j \mid 1 \leq j < m\}$  (respectively,  $\{F_j \mid 1 \leq j < n\}$ ) be a set of orthogonal abelian projections with the following properties: (1)  $A_j \neq 0$  (respectively,  $B_j \neq 0$ ) for all  $j$ ; (2)  $A_1 \geq A_2 \geq \dots$  (respectively,  $B_1 \geq B_2 \geq \dots$ ); (3) if  $Z$  is the spectrum of the center, then*

$$\{\zeta \in Z \mid E_j(\zeta) \neq 0\} = \text{closure } \{\zeta \in Z \mid A_j^\wedge(\zeta) \neq 0\}$$

*(respectively,  $\{\zeta \in Z \mid F_j(\zeta) \neq 0\} = \text{closure } \{\zeta \in Z \mid B_j^\wedge(\zeta) \neq 0\}$ ) for every  $j$ ; (4) if  $m = +\infty$  (respectively,  $n = +\infty$ ), then  $\lim_j A_j = 0$  (respectively,  $\lim_j B_j = 0$ ); (5)  $\sum A_j E_j = A$  (respectively,  $\sum B_j F_j = A$ ). Then  $m = n$  and  $A_j = B_j$  for every  $j$ .*

**Proof.** For every  $\zeta$  in  $Z$  the element  $A(\zeta)$  is a completely continuous operator on  $H(\zeta)$ . We have that  $\sum A_j^\wedge(\zeta)E_j(\zeta) = A(\zeta) = \sum B_j^\wedge(\zeta)F_j(\zeta)$  on  $H(\zeta)$  since for both sums  $A$  is the uniform limit of the partial sums  $\sum \{A_j E_j \mid 1 \leq j \leq n\}$  and  $\sum \{B_j F_j \mid 1 \leq j \leq n\}$ . However,  $\{A_j^\wedge(\zeta)\}$  and  $\{B_j^\wedge(\zeta)\}$  are decreasing sets of positive numbers and the nonzero  $E_j(\zeta)$  and  $F_j(\zeta)$  are one-dimensional projections on  $H(\zeta)$ . Because  $E_j(\zeta) = 0$  (respectively,  $F_j(\zeta) = 0$ ) implies  $E_k(\zeta) = 0$  (respectively,  $F_k(\zeta) = 0$ ) whenever  $j \leq k$ , we have by the spectral theorem on  $H(\zeta)$  that  $A_j^\wedge(\zeta) \neq 0$  if and only if  $B_j^\wedge(\zeta) \neq 0$  and that  $A_j^\wedge(\zeta) = B_j^\wedge(\zeta)$  for all  $j$  such that  $A_j^\wedge(\zeta) \neq 0$ . Thus, we have the result that  $m = n$  and  $A_j^\wedge(\zeta) = B_j^\wedge(\zeta)$  for all  $\zeta \in Z$ . Thus,  $A_j = B_j$  for all  $j$ . Q.E.D.

We call the representation for a self-adjoint operator  $A$  in Theorem 2.2 a spectral representation for  $A$ .

**3. Elements with continuous trace.** Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $P(\mathcal{A})$  be the set of all primitive ideals of  $\mathcal{A}$  with the hull-kernel topology. The space  $P(\mathcal{A})$  is called the structure space of  $\mathcal{A}$ . Let  $\mathcal{A}^\wedge$  be the set of all equivalence classes of irreducible representations of  $\mathcal{A}$  with the reciprocal topology induced by the function  $\phi \rightarrow \text{kernel } \phi$  of  $\mathcal{A}^\wedge$  into  $P(\mathcal{A})$ . For each  $A$  in  $\mathcal{A}^+$  the trace of the image of  $A$  under any

of the representations in a single equivalence class  $\phi$  in  $\mathcal{A}^\wedge$  is invariant; the trace is denoted by  $\text{Tr}(\phi(A))$ . If  $A$  is an element in  $\mathcal{A}^+$  such that  $\text{Tr}(\phi(A))$  is finite for all  $\phi \in \mathcal{A}^\wedge$  and such that  $\phi \rightarrow \text{Tr}(\phi(A))$  is continuous on  $\mathcal{A}^\wedge$ , then  $A$  is said to have a continuous trace. The set of all positive elements with continuous trace is the set of all positive elements of a two-sided ideal in  $m(\mathcal{A})$  of  $\mathcal{A}$  [3, §4]. In this section we study the elements with continuous trace in a von Neumann algebra.

We first prove a theorem concerning the structure space of  $I_a$ .

**THEOREM 3.1.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $\mathcal{Z}$  be the center of  $\mathcal{A}$ ; let  $Z$  be the spectrum of  $\mathcal{Z}$ . The structure space  $P_a = P(I_a)$  is homeomorphic to  $Z$  under the map  $f(\zeta) = [\zeta] \cap I_a$ .*

**Proof.** The set  $P_a$  is equal to  $\{[\zeta] \cap I_a \mid \zeta \in Z\}$ . Therefore  $f$  is a surjection. We prove that  $f$  is an injection and that  $f$  is bicontinuous.

Suppose  $\zeta_1$  and  $\zeta_2$  are distinct elements in  $Z$ . There is a projection  $P$  in  $\mathcal{Z}$  such that  $P^\wedge(\zeta_1) = 1$  and  $P^\wedge(\zeta_2) = 0$ . Let  $E$  be an abelian projection of central support  $P$ ; then  $E(\zeta_1) \neq 0$  and  $E(\zeta_2) = 0$ . Now for any  $A$  in  $I_a$  and any  $\zeta \in Z$  we have that  $\|A(\zeta)\| = \|A([\zeta] \cap I_a)\|$ ; thus,  $E \notin f(\zeta_1)$  and  $E \in f(\zeta_2)$  so  $f(\zeta_1) \neq f(\zeta_2)$ . This shows that  $f$  is an injection.

We now show that  $f$  is continuous. Let  $X$  be an open set in  $P_a$ . Then there is a closed two-sided ideal  $I$  in  $I_a$  such that  $X = \{J \in P_a \mid J \not\supset I\}$ . We show that the set  $\{\zeta \in Z \mid f(\zeta) \in X\}$  is open in  $Z$ . Indeed, we have that  $f(\zeta) \not\supset I$  if and only if  $[\zeta] \not\supset I$ . But if  $[\zeta_0] \supset I$  there is a projection  $F$  in  $I$  such that  $\|F(\zeta_0)\| = 1$ . The set  $\{\zeta \in Z \mid \|F(\zeta)\| = 1\}$  is open in  $Z$  and contains  $[\zeta_0]$  and so  $\{\zeta \in Z \mid f(\zeta) \not\supset I\}$  is open in  $P_a$ . Thus  $f$  is continuous.

Let  $Y$  be a closed set in  $Z$ . The set  $Y$  is compact and, therefore,  $f(Y)$  is compact in the Hausdorff space  $P_a$ . So  $f(Y)$  is closed in  $P_a$ . This proves  $f$  is closed and that  $f^{-1}$  is continuous. Q.E.D.

Because  $I_a$  is a CCR algebra, the space  $I_a^\wedge$  is homeomorphic to  $P_a$  under the mapping  $\phi \rightarrow \text{kernel } \phi$ . If  $\psi$  is an irreducible representation whose equivalence class is  $(\psi)$  the function  $\zeta \rightarrow (\psi_\zeta)$  is a homeomorphism of  $Z$  onto  $I_a^\wedge$ .

We are now ready to characterize  $m(I_a)^+$ . Let  $\mathcal{Q}$  be the set of all positive elements  $A$  in  $I_a$  such that if  $\sum A_j E_j$  is a spectral resolution of  $A$  then  $\sum A_j$  converges uniformly in  $\mathcal{Z}$ . In particular  $\mathcal{Q}$  contains every projection of  $I_a$ . By Theorem 2.3 the sum  $\sum A_j$  depends only on  $A$ . Define the trace  $\text{Tr}(A)$  of  $A$  to be  $\sum A_j$ .

**THEOREM 3.2.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $\mathcal{Z}$  be the center of  $\mathcal{A}$ . Let  $I_a$  be the ideal in  $\mathcal{A}$  generated by the abelian projections of  $\mathcal{A}$ . The set  $m(I_a)^+$  of positive elements of  $I_a$  with continuous trace is equal to the set  $\mathcal{Q}$ . Furthermore, if  $A \in \mathcal{Q}$ ,  $\text{Tr}(A)^\wedge(\zeta) = \text{Tr}(\psi_\zeta(A))$  for each  $\zeta$  in the spectrum  $Z$  of  $\mathcal{Z}$ .*

**Proof.** Let  $A$  be an element in  $\mathcal{Q}$  and let  $\sum A_j E_j$  be a spectral representation of  $A$ . For each  $\phi = (\psi_\zeta)$  in  $I_a^\wedge$  where  $\zeta \in Z$  we see that  $\text{Tr}(\phi(A)) = \sum A_j^\wedge(\zeta) = \text{Tr}(A)^\wedge(\zeta)$ . Since  $Z$  is homeomorphic to  $I_a^\wedge$  under the mapping  $\zeta \rightarrow (\psi_\zeta)$ ,  $\phi \rightarrow \text{Tr}(\phi(A))$  is

continuous on  $I_a^\wedge$ . Indeed, this function is the uniform limit of continuous functions. So  $\mathcal{Q} \subset m(I_a)^+$ .

Conversely, suppose that  $A \in m(I_a)^+$  and let  $\sum A_j E_j$  be a spectral representation of  $A$ . Then the function  $\zeta \rightarrow \sum A_j^\wedge(\zeta) = g(\zeta)$  is continuous on  $Z$  because  $\text{Tr}(\psi_\zeta(A)) = \sum A_j^\wedge(\zeta)$ . For each  $j$  let us assume  $A_j \neq 0$ . Then if  $g_k = \sum \{A_j^\wedge \mid 1 \leq j \leq k\}$ , the sequence  $\{g_k\}$  of monotonically increasing continuous functions converges pointwise to the continuous function  $g$ . By Dini's theorem  $g$  is the uniform limit of the  $g_k$  i.e.  $\sum A_j$  converges uniformly. So  $A \in \mathcal{Q}$ . This proves  $m(I_a)^+ \subset \mathcal{Q}$ .

**COROLLARY.** For each  $A_1, A_2 \in \mathcal{Q}$  and  $C_1, C_2 \in \mathcal{Z}^+$   $C_j A_j$  ( $j=1, 2$ ) are elements of  $\mathcal{Q}$  and  $\text{Tr}(C_1 A_1 + C_2 A_2) = C_1 \text{Tr}(A_1) + C_2 \text{Tr}(A_2)$ .

**Proof.** It is obvious that  $C_j A_j \in \mathcal{Q}$ . Also for each  $\zeta \in Z$

$$\begin{aligned} \text{Tr}(C_1 A_1 + C_2 A_2)^\wedge(\zeta) &= \text{Tr}(C_1^\wedge(\zeta) A_1(\zeta) + C_2^\wedge(\zeta) A_2(\zeta)) \\ &= C_1^\wedge(\zeta) \text{Tr}(A_1(\zeta)) + C_2^\wedge(\zeta) \text{Tr}(A_2(\zeta)) \\ &= (C_1 \text{Tr}(A_1) + C_2 \text{Tr}(A_2))^\wedge(\zeta). \end{aligned}$$

Thus,  $\text{Tr}(\sum C_j A_j) = \sum C_j \text{Tr}(A_j)$ . We show that  $m(I_a)$  is an ideal in  $\mathcal{A}$ .

**THEOREM 3.3.** The ideal  $m(I_a)$  of elements in  $I_a$  with continuous trace is an ideal in  $\mathcal{A}$ .

**Proof.** If  $A \in \mathcal{A}$  and  $A^* A \in I_a$  then  $A \in I_a$ . Thus,  $J = \{A \in I_a \mid A^* A \in \mathcal{Q}\} = \{A \in \mathcal{A} \mid A^* A \in \mathcal{Q}\}$ . The set  $\mathcal{Q}$  has the following properties: (1)  $\mathcal{Q} + \mathcal{Q} \subset \mathcal{Q}$ ; (2) if  $A \in \mathcal{A}$  and  $A^* A \in \mathcal{Q}$  then  $AA^* \in \mathcal{Q}$ ; and (3) if  $B \in \mathcal{Q}$ ,  $A \in \mathcal{A}$  and  $0 \leq A \leq B$ , then  $A \in \mathcal{Q}$ . So  $J$  is a two-sided ideal in  $\mathcal{A}$  and  $m(I_a) = J^2$  is an ideal in  $\mathcal{A}$ . Q.E.D.

In the standard way  $\text{Tr}$  on  $\mathcal{Q}$  may be extended to a function of  $m(I_a)$  into  $\mathcal{Z}$  such that  $\text{Tr}(C_1 A_1 + C_2 A_2) = C_1 \text{Tr}(A_1) = C_2 \text{Tr}(A_2)$  for every  $A_1, A_2 \in m(I_a)$  and  $C_1, C_2 \in \mathcal{Z}$ . For each  $\zeta \in Z$  and  $A \in m(I_a)$  it is true that  $\text{Tr}(A)^\wedge(\zeta) = \text{Tr}(A(\zeta))$ .

**LEMMA.** Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$ . Let  $Z$  be the spectrum of  $\mathcal{Z}$  and let  $\zeta \in Z$ . Let  $\{x_n\}$  be a sequence of orthonormal vectors in  $H(\zeta)$ . There is a sequence of abelian projections  $\{E_n\}$  in  $\mathcal{A}$  such that  $E_n \succ E_{n+1}$  and  $E_n(\zeta)x_n = x_n$  for all  $n$ .

**Proof.** This lemma is a simple extension of a result of Glimm [15, §5, Lemma 11]. Let  $E_1$  be a projection in the ideal  $I_a$  generated by the abelian projections such that  $E_1(\zeta)x_1 = x_1$ . Indeed,  $I_a(\zeta)$  is the ideal of completely continuous operators on  $H(\zeta)$ . Suppose  $E_1, E_2, \dots, E_n$  have been constructed. Let  $F$  be a projection in  $I_a$  such that  $F(\zeta)$  is the projection of  $H(\zeta)$  onto the subspace spanned by  $x_{n+1}$  and such that  $(E_1 + E_2 + \dots + E_n)(\zeta)F(\zeta) = 0$  in an open and closed neighborhood  $W$  of  $\zeta$ . Let  $F'$  be an abelian projection whose central support  $R$  is the same as the central support of  $F$ . Then  $F'(\zeta) \neq 0$  and so  $F'(\zeta)$  is the projection of  $H(\zeta)$  on the subspace

spanned by  $x_{n+1}$ . Let  $P$  be the central support of  $E_n$  and let  $Q$  be the central projection corresponding to  $W$ . Then let  $E_{n+1} = F'PQ$ . We have that  $E_{n+1}$  is abelian and  $E_{n+1}(\zeta) = F'(\zeta)$ . Also  $E_j E_{n+1} = 0$  for  $j = 1, 2, \dots, n$  and  $E_n \succ E_{n+1}$  because the central support of  $E_n$  majorizes that of  $E_{n+1}$ . Q.E.D.

**THEOREM 3.4.** *Let  $m(I_a)$  be the ideal of elements with continuous trace in the ideal generated by the abelian projections of a Type I von Neumann algebra  $\mathcal{A}$ . For every  $\zeta$  in the spectrum  $Z$  of the center of  $\mathcal{A}$  the image  $m(I_a)(\zeta)$  of  $m(I_a)$  under the homeomorphism  $\psi_\zeta$  of  $\mathcal{A}$  with kernel  $[\zeta]$  is the trace class of  $H(\zeta)$ .*

**Proof.** Let  $A \in \mathcal{Q}$ . Then  $\psi_\zeta(A)$  ( $\zeta \in Z$ ) has a trace on  $H(\zeta)$ . So  $A(\zeta)$  is of trace class on  $H(\zeta)$ . Thus  $m(I_a)(\zeta)$  is contained in the trace class of  $H(\zeta)$ .

Now let  $A$  be in the trace class of  $H(\zeta)$ . We may assume  $A$  is positive. There is a sequence  $\{x_j\}$  of orthonormal vectors in  $H(\zeta)$  and a decreasing sequence of positive real numbers such that  $A = \sum \alpha_j P_j$  where  $P_j$  is the one-dimensional projection on the subspace spanned by  $x_j$ . There is a sequence of mutually orthogonal abelian projections  $\{E_j\}$  in  $\mathcal{A}$  such that  $E_j \succ E_{j+1}$  (all  $j$ ) and  $E_j(\zeta) = P_j$ . Then the operator  $\sum \alpha_j E_j = B$  is in  $I_a$  and if  $R_j$  is the central support of  $E_j$ ,  $\sum (\alpha_j R_j) E_j$  is a spectral resolution for  $B$ . But  $\sum \|\alpha_j R_j\| = \sum \alpha_j < +\infty$  and so  $B \in \mathcal{Q}$ . Furthermore,  $B(\zeta) = A$ . This proves  $m(I_a)(\zeta)$  contains the trace class of  $H(\zeta)$ . Hence,  $m(I_a)(\zeta)$  is the trace class of  $H(\zeta)$ . Q.E.D.

We now determine the set  $m(\mathcal{A})^+$  when  $\mathcal{A}$  is a von Neumann algebra.

**THEOREM 3.5.** *Let  $\mathcal{A}$  be a von Neumann algebra with center  $\mathcal{Z}$ . For each  $n = 1, 2, \dots$  let  $R_n$  be the largest projection  $P$  in  $\mathcal{Z}$  such that  $\mathcal{A}P$  is of Type  $1_n$ . Let  $R = \sum R_n$  and let  $\mathcal{Q}R = \{A \in \mathcal{Q} \mid A(1 - R) = 0\}$ . Then  $m(\mathcal{A})^+$  is equal to the set of all  $A$  in  $\mathcal{Q}R$  such that  $\lim_n \text{Tr}(A)R_n = 0$ .*

**Proof.** The uniform closure  $I$  of  $m(\mathcal{A})$  is a uniformly closed two-sided ideal in  $\mathcal{A}$ ; the space  $I^\wedge$  is homeomorphic to the open subset  $\{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \nsubseteq I\}$  under the inverse of the map  $(\psi) \rightarrow (\psi|I)$ . Therefore  $\phi \rightarrow \text{Tr}(\phi(A))$  is continuous on  $I^\wedge$  for all  $A \in m(\mathcal{A})^+$ . This shows that  $I$  is a CCR ideal in  $\mathcal{A}$  [3, §4]. Because  $I_a$  is the maximal GCR ideal in  $\mathcal{A}$ , we have that  $I \subset I_a$ . So if  $P$  is the central projection of  $\mathcal{A}$  such that  $\mathcal{A}P$  is discrete and  $\mathcal{A}(1 - P)$  is continuous we have that  $m(\mathcal{A}) \cdot (1 - P) = 0$ .

Now we show that  $m(\mathcal{A})(P - R) = (0)$ . Let  $A \in m(\mathcal{A})^+$ . Assume  $P - R \neq 0$ . Let  $Z$  be the spectrum of  $\mathcal{Z}$  and let  $X = \{\zeta \in Z \mid (P - R)^\wedge(\zeta) = 1\}$ . For each  $\zeta \in X$  the ideal  $I_a + [\zeta]$  is proper. Indeed, the representation  $\psi_\zeta$  is an irreducible representation of  $\mathcal{A}$  on  $H(\zeta)$  such that  $\psi_\zeta(I_a)$  is the ideal of completely continuous operators on  $H(\zeta)$ . But for any  $n = 1, 2, \dots$ ,  $P - R$  may be written as the sum of  $n$  equivalent projections; so  $\psi_\zeta(1)$  does not have finite dimensions in  $H(\zeta)$  and thus  $\psi_\zeta(1) \notin \psi_\zeta(I_a)$ . There is therefore an element  $\phi \in \mathcal{A}^\wedge$  such that  $\text{kernel } \phi \supset I_a + [\zeta]$ . Let  $U$  be a neighborhood of  $\phi$  in  $\mathcal{A}^\wedge$ . There is an ideal  $I'$  in  $\mathcal{A}$  such that  $U = \{\phi' \in \mathcal{A}^\wedge \mid \text{kernel } \phi' \nsubseteq I'\}$ ; so  $\text{kernel } \phi \nsubseteq I'$  and thus  $\text{kernel } (\psi_\zeta) = [\zeta] \nsubseteq I'$ . This shows that  $(\psi_\zeta) \in U$ . So a net each of whose terms is  $(\psi_\zeta)$  converges to  $\phi$  in  $\mathcal{A}^\wedge$ . But  $\text{Tr}(\phi(A)) = 0$  since



$m(\mathcal{A})^+ \subset I_a$ . If  $\sum A_j E_j$  is a spectral representation of  $A$ , then  $\psi_\zeta(A) = \sum_j A_j^\wedge(\zeta) E_j(\zeta)$  and so  $\text{Tr}(\psi_\zeta(A)) = \sum_j A_j^\wedge(\zeta) < +\infty$ . This shows that  $\sum A_j^\wedge(\zeta) = \text{Tr}(\phi(A)) = 0$  and that  $A(\zeta) = 0$ . We have therefore that  $A(P-R) = 0$  and consequently that  $m(\mathcal{A})(P-R) = 0$ .

Since  $I_a^\wedge$  is homeomorphic to the open subset  $\{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset I_a\}$  under inverse of the map  $(\psi) \rightarrow (\psi|I_a)$ , the function  $\phi \rightarrow \text{Tr}(\phi(A))$  is continuous on  $I_a^\wedge$  for each  $A \in m(\mathcal{A})^+$ . This means that  $m(\mathcal{A})^+ \subset \mathcal{Q}R$ . Let  $A \in m(\mathcal{A})^+$ . We show that  $\lim_n \text{Tr}(A)R_n = 0$ . Arguing by contradiction we assume that there is an  $\varepsilon > 0$  such that  $\|\text{Tr}(A)R_n\| \geq \varepsilon$  for an infinite number of  $n$ . There is no loss in generality in assuming that  $\|\text{Tr}(A)R_n\| \geq \varepsilon$  for every  $n$ , since we may replace  $\{\text{Tr}(A)R_n\}$  by a subsequence if necessary. For each  $n$  there is a  $\zeta_n$  in the set  $Y_n = \{\zeta \in Z \mid R_n^\wedge(\zeta) = 1\}$  such that  $(\text{Tr}(A)R_n)^\wedge(\zeta_n) \geq \varepsilon/2$ . There is a cluster point  $\zeta_0$  for the set  $\{\zeta_n \mid n = 1, 2, \dots\}$ . We have that  $\zeta_0 \in Y - \bigcup_n Y_n$  where  $Y = \{\zeta \in Z \mid R^\wedge(\zeta) = 1\}$ . However,  $I_a + [\zeta_0]$  is a proper ideal in  $\mathcal{A}$ . Let  $\phi \in \mathcal{A}^\wedge$  be such that  $\text{kernel } \phi \supset I_a + [\zeta_0]$ . Every open neighborhood of  $\phi$  in  $\mathcal{A}^\wedge$  contains  $(\psi_{\zeta_0})$ . Since  $Z$  is homeomorphic to  $I_a^\wedge = P_a$  and since  $I_a^\wedge$  is an open subset of  $\mathcal{A}^\wedge$ ,  $(\psi_{\zeta_0})$  is a cluster point of  $\{(\psi_{\zeta_n}) \mid n = 1, 2, \dots\}$  in  $\mathcal{A}^\wedge$ . So there is a subset of  $\{\text{Tr}(\psi_{\zeta_n}(A))\}$  converging to  $\text{Tr}(\phi(A))$ . This is impossible since  $\text{Tr}(\phi(A)) = 0$  and  $\text{Tr}(\psi_{\zeta_n}(A)) \geq \varepsilon/2$  for all  $n$ . Therefore,  $\lim_n \text{Tr}(A)R_n = 0$ .

Now suppose that  $A \in \mathcal{Q}R$  and  $\lim_n \text{Tr}(A)R_n = 0$ . Since the set  $\{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset \mathcal{A}(1-R)\} = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \supset \mathcal{A}R\}$  is open and closed and since  $\text{Tr}(\phi(A)) = 0$  for all  $\phi$  in this set, it is sufficient to prove that  $\phi \rightarrow \text{Tr}(\phi(A))$  is finite and continuous on  $\{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \supset \mathcal{A}(1-R)\} = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset \mathcal{A}R\} = S$ . Let  $\phi_0 \in S$ ; there is a unique  $\zeta_0 \in Y$  such that  $\text{kernel } \phi_0 \supset [\zeta_0]$ . Let  $\varepsilon > 0$ .

If  $\zeta_0 \in Y_n$  for some  $n$ , then  $(\psi_{\zeta_0}) = \phi_0$ . There is a projection  $Q$  in  $\mathcal{Z}$  such that  $Q \leq R_n$  such that  $Q^\wedge(\zeta_0) = 1$  and  $|\text{Tr}(A)^\wedge(\zeta_0) - \text{Tr}(A)^\wedge(\zeta)| < \varepsilon$  whenever  $Q^\wedge(\zeta) = 1$ . For every  $\phi$  in the open set  $\{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset \mathcal{A}Q\}$ , there is a  $\zeta \in Z$  such that  $Q^\wedge(\zeta) = 1$  and  $(\psi_\zeta) = \phi$ . Indeed, there is a  $\zeta \in Z$  such that  $\text{kernel } \phi \supset [\zeta]$ ; so  $\mathcal{A}Q$  is not contained in  $[\zeta]$  and  $Q^\wedge(\zeta) = 1$ . This shows that  $\mathcal{A}(\zeta)$  is the algebra of all linear operators on an  $n$ -dimensional Hilbert space and so  $\phi = (\psi_\zeta)$ . So if  $\phi \in \mathcal{A}^\wedge$  and  $\text{kernel } \phi \not\supset \mathcal{A}Q$  then  $\phi = (\psi_\zeta)$  for some  $\zeta$  such that  $Q^\wedge(\zeta) = 1$ ,  $\text{Tr}(\phi(A)) = \text{Tr}(A)^\wedge(\zeta)$ , and  $|\text{Tr}(\phi_0(A)) - \text{Tr}(\phi(A))| = |\text{Tr}(A)^\wedge(\zeta_0) - \text{Tr}(A)^\wedge(\zeta)| < \varepsilon$ .

If on the other hand  $\zeta_0 \in Y - \bigcup_n Y_n$  then  $\text{Tr}(\phi_0(A)) = 0$  because  $A(\zeta_0) = 0$ . There is an integer  $N$  such that  $\|\text{Tr}(A)(1 - \sum \{R_n \mid 1 \leq n \leq N\})\| < \varepsilon$ . The open set  $V = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset \mathcal{A}(1 - \sum \{R_n \mid 1 \leq n \leq N\})\}$  can be written as the union of three disjoint sets  $S_1, S_2, S_3$  where  $S_1 = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \not\supset \mathcal{A}(1-R)\}$ ,  $S_2 = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \supset [\zeta] \text{ for some } \zeta \text{ in } Y - \bigcup_n Y_n\}$  and  $S_3 = \{\phi \in \mathcal{A}^\wedge \mid \text{kernel } \phi \supset [\zeta] \text{ for some } \zeta \text{ in } Y_{N+j}, j = 1, 2, \dots\}$ . So we have that  $\phi(A) = 0$  whenever  $\phi \in S_1$  and  $\phi(A) = 0$  (because  $A(\zeta) = 0$ ) whenever  $\phi \in S_2$ . Finally,  $\text{Tr}(A)^\wedge(\zeta) \leq \varepsilon$  whenever  $\zeta \in Y_{N+j}$  ( $j = 1, 2, \dots$ ) and so  $\text{Tr}(\phi(A)) = \text{Tr}(A)^\wedge(\zeta) \leq \varepsilon$  whenever  $\phi \in S_3$  and  $\text{kernel } \phi \supset [\zeta]$ . This shows that  $\text{Tr}(\phi(A)) \leq \varepsilon$  on an open set  $V$  containing  $\phi_0$ . Therefore, the function  $\phi \rightarrow \text{Tr}(\phi(A))$  is continuous on  $\mathcal{A}^\wedge$ . Q.E.D.

4. **The Schmidt class and the trace class of a Type I von Neumann algebra.** Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$ ; let  $I_a$  be the ideal generated by the abelian projections. Let  $\mathcal{P}$  be the set of all positive elements  $A$  in  $I_a$  such that if  $\sum A_j E_j$  is a spectral representation for  $A$  then the increasing sequence  $\{\sum (A_j \mid 1 \leq j \leq n)\}_n$  is strongly convergent in  $\mathcal{Z}$ . Thus,  $A \in \mathcal{P}$  if and only if  $\{\sum (A_j \mid 1 \leq j \leq n)\}$  is bounded above. The set  $\mathcal{P}$  will serve as the starting point for the construction of the Schmidt class and the trace class.

If  $A \in I_a^+$  and if  $\zeta \rightarrow \text{Tr}(A(\zeta))$  is defined in a neighborhood of the point  $\zeta_0$  in the spectrum of  $\mathcal{Z}$  and is continuous at  $\zeta_0$ , then there is a central projection  $P$  with  $P \wedge (\zeta_0) = 1$  and  $AP \in \mathcal{P}$ . Indeed, if  $\sum A_j E_j$  is a resolution of  $A$  for all  $\zeta$  in some open and closed neighborhood  $U$  of  $\zeta_0$  we have that  $\text{lub}_n \sum \{A_j \wedge (\zeta) \mid 1 \leq j \leq n\} \leq \text{Tr}(A(\zeta_0)) + 1$ . Thus  $\{\sum (A_j P \mid 1 \leq j \leq n)\}_n$  is bounded above in  $\mathcal{Z}$ . Here  $P$  is the projection which corresponds to  $U$ .

The relation of  $\mathcal{P}$  and  $\mathcal{Q}$  (§3) is clarified by the following theorem based on a result of von Neumann [10].

**THEOREM 4.1.** *Let  $A$  be an element of the set  $\mathcal{P}$ . There is a set  $S$  of mutually orthogonal central projections of least upper bound 1 such that  $PA \in \mathcal{Q}$  for each  $P \in S$ .*

**Proof.** Let  $\sum A_j E_j$  be a spectral resolution for  $A$ . It is sufficient to show that there is a nonzero projection  $P$  in  $\mathcal{Z}$  such that  $\{\sum (A_j P \mid 1 \leq j \leq n)\}_n$  converges uniformly. Let  $\mathcal{B}$  be the  $C^*$ -algebra generated by  $\{A_j \mid j=1, 2, \dots\} \cup \{1\}$ . The algebra  $\mathcal{B}$  is separable. Let  $H$  be the Hilbert space of  $\mathcal{A}$  and let  $x$  be a unit vector of  $H$ . The subspace  $K = \text{closure}\{Bx \mid B \in \mathcal{B}\}$  is separable and the projection  $E'$  corresponding to  $K$  is an element of the commutator  $\mathcal{B}'$  of  $\mathcal{B}$  on  $H$ . If  $E$  is the central support of  $E'$ , then  $E$  is an element of the commutative von Neumann algebra  $\mathcal{B}''$  generated by  $\mathcal{B}$  on  $H$ . Also  $\mathcal{B}'' \subset \mathcal{Z}$  because  $\mathcal{B}''$  is the weak closure of  $\mathcal{B}$  and  $\mathcal{B}''E$  is isomorphic to the von Neumann algebra  $\mathcal{B}''E'$  on  $E'(H)$ . Let  $B_n = \sum \{A_j \mid 1 \leq j \leq n\} E'$ ,  $n=1, 2, \dots$ . Then  $\{B_n\}$  converges strongly to some  $B$  in  $\mathcal{B}''E'$ . There is a subsequence  $\{B_{n_k}\}$  of  $\{B_n\}$  and an element  $C \in \mathcal{B}''E'$  such that (1)  $Cy=0$  implies  $y=0$  for  $y \in E'(H)$  and (2)  $E' \geq \sum \{((B_{n_k} - B)C)^*((B_{n_k} - B)C) \mid 1 \leq k \leq m\}$  for all  $m=1, 2, \dots$ . So

$$E' \geq C^*C \sum \{(B_{n_k} - B)^*(B_{n_k} - B) \mid 1 \leq k \leq m\}$$

for all  $m=1, 2, \dots$ . There is a nonzero projection  $F \in \mathcal{B}''E'$  such that  $FC^*C \geq 2^{-1}F\|C\|^2$ . Therefore, there is an  $\alpha > 0$  such that

$$\alpha F \geq \sum \{(B_{n_k} - B)^*(B_{n_k} - B) \mid 1 \leq k \leq m\}F$$

for all  $m=1, 2, \dots$ . For all  $\zeta$  in the spectrum  $X$  of  $\mathcal{B}''E'$  we have that

$$\lim_k B_{n_k} \wedge (\zeta) F \wedge (\zeta) = B \wedge (\zeta) F \wedge (\zeta).$$

Because  $\{(B_{n_k} F) \wedge (\zeta)\}_k$  is a monotonically increasing sequence of continuous functions on the compact set  $X$ , we have  $\lim_k B_{n_k} F = BF$  (Dini's Theorem). So  $\lim_n B_n F = BF$

(uniformly). Because  $\mathcal{B}''E$  is isomorphic to  $\mathcal{B}''E'$  there is a projection  $P$  in  $\mathcal{Z}$  such that  $\{\sum (A_j P \mid 1 \leq j \leq n)\}$  converges uniformly. Q.E.D.

Let  $F$  be an abelian projection in the Type I algebra  $\mathcal{A}$  with center  $\mathcal{Z}$  and let  $P$  be the central support of  $F$ . The algebra  $F\mathcal{A}F$  is isomorphic to  $\mathcal{Z}P$ . We let  $\tau_F(A)$  denote the unique element in  $\mathcal{Z}P$  such that  $FAF = \tau_F(A)F$ .

We make use of the following lemma.

**LEMMA.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $E$  and  $F$  be abelian projections. Then  $\tau_E(F) = \tau_F(E)$ .*

**Proof.** We have  $FEFEF$  is equal to both  $\tau_E(F)\tau_F(E)F$  and  $(\tau_F(E))^2F$ ; similarly,  $\tau_E(F)\tau_F(E)E = (\tau_E(F))^2E$ . If  $P$  (respectively  $Q$ ) is the central support of  $E$  (respectively  $F$ ), the elements  $\tau_E(F)$  and  $\tau_F(E)$  are majorized by  $PQ$ . Let  $\zeta$  be an element in the spectrum of the center of  $\mathcal{A}$  such that  $(PQ)^\wedge(\zeta) = 1$ . Then  $F(\zeta) \neq 0$  and  $E(\zeta) \neq 0$ , and  $\tau_E(F)^\wedge(\zeta)^2 = \tau_E(F)^\wedge(\zeta)\tau_F(E)^\wedge(\zeta) = \tau_F(E)^\wedge(\zeta)^2$ . So  $\tau_E(F)^\wedge(\zeta) = \tau_F(E)^\wedge(\zeta)$ ; this shows that  $\tau_E(F)^\wedge(\zeta) = \tau_F(E)^\wedge(\zeta)$  for all  $\zeta$  in the spectrum of the center. Thus,  $\tau_E(F) = \tau_F(E)$ .

**PROPOSITION 4.2.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $I_a$  be the ideal generated by the abelian projections. Let  $A$  be a positive element in  $\mathcal{A}$ , and let  $S$  be a set of mutually orthogonal abelian projections. Let  $F(S)$  be the set of all finite subsets of  $S$ . Let  $N(A, S)$  be the increasing net in  $\mathcal{Z}^+$  given by*

$$\left\{ \sum \{ \tau_F(A) \mid F \in \pi \} \mid \pi \in F(S) \right\}.$$

*Then  $A$  is an element of  $\mathcal{P}$  if and only if  $N(A, S)$  is bounded above for every set  $S$  of mutually orthogonal abelian projections.*

**Proof.** Let  $A$  be an element of  $\mathcal{P}$  and let  $\sum A_j E_j$  be a spectral resolution of  $A$ . Let  $\sum A_j$  denote the least upper bound of  $\{\sum (A_j \mid 1 \leq j \leq n)\}_n$ . Let  $S$  be a set of mutually orthogonal abelian projections and let  $F(S)$  be the set of all finite subsets of  $S$ . Then if  $\pi \in F(S)$ ,

$$\begin{aligned} \sum \{ \tau_F(A) \mid F \in \pi \} &= \sum \left\{ \tau_F \left( \sum_j A_j E_j \right) \mid F \in \pi \right\} \\ &= \sum_j A_j \tau_{E_j} \left( \sum \{ F \mid F \in \pi \} \right) \leq \sum A_j \end{aligned}$$

because  $\lim_n \tau_F(\sum \{ A_j E_j \mid 1 \leq j \leq n \}) = \tau_F(A)$  uniformly. Indeed,  $\tau_F$  is an isometry of  $F\mathcal{A}F$  onto  $\mathcal{Z}P$  ( $P$  central support of  $F$ ) and  $\lim_n \sum \{ A_j E_j \mid 1 \leq j \leq n \} = A$  uniformly. This proves that  $\{\sum \{ \tau_F(A) \mid F \in \pi \} \mid \pi \in F(S)\}$  is bounded above by  $\sum A_j$ .

Conversely, suppose that  $A$  is a positive element in  $\mathcal{A}$  and that for every set  $S$  of mutually orthogonal abelian projections the set  $\{\sum \{ \tau_F(A) \mid F \in \pi \} \mid \pi \in F(S)\}$  is bounded above. We show that  $A$  is an element of  $\mathcal{P}$ . It is sufficient to show  $A$  is an element of  $I_a$ . If  $\varepsilon > 0$  is given we find an element  $B \in I_a$  such that  $\|B - A\| \leq \varepsilon$ . Let  $Z$  be the spectrum of the center of  $\mathcal{A}$ . Suppose  $\zeta \in Z$ ; we show  $A(\zeta) \in I_a(\zeta)$ . Indeed,

if  $A(\zeta) \notin I_a(\zeta)$ , there is a sequence  $x_j$  of orthonormal vectors in  $H(\zeta)$  such that the set  $(\sum \{(A(\zeta)x_j, x_j) \mid 1 \leq j \leq n\})_n$  is not bounded above because  $I_a(\zeta)$  is the set of all completely continuous operators on  $H(\zeta)$ . Now there is a sequence  $\{F_j\}$  of orthogonal abelian projections in  $\mathcal{A}$  such that  $F_j(\zeta)$  is the one-dimensional projection on the subspace generated by  $x_j$ . We have that  $(\sum \{\tau_{F_j}(A) \mid 1 \leq j \leq n\})_n$  is not bounded above in the center because  $(\sum \{\tau_{F_j}(A) \mid 1 \leq j \leq n\})^\wedge(\zeta) = \sum \{(A(\zeta)x_j, x_j) \mid 1 \leq j \leq n\}$  is not bounded above. This is a contradiction. So we conclude that  $A(\zeta) \in I_a(\zeta)$  for every  $\zeta$  in the spectrum of the center. Since  $\zeta \rightarrow \|C(\zeta)\|$  ( $C \in \mathcal{A}$ ) is continuous on the spectrum  $Z$  there are a finite number  $\{U_j \mid 1 \leq j \leq m\}$  of disjoint open and closed sets and a corresponding number  $\{B_j \mid 1 \leq j \leq m\}$  of elements in  $I_a$  such that  $\|A(\zeta) - B_j(\zeta)\| < \varepsilon$  for  $\zeta \in U_j$  and  $1 \leq j \leq m$ . If  $P_j$  ( $1 \leq j \leq m$ ) is the projection of  $\mathcal{Z}$  corresponding to  $U_j$  ( $1 \leq j \leq m$ ), then  $B = \sum B_j P_j \in I_a$  and  $\|A - B\| \leq \varepsilon$ . Q.E.D.

Let  $A$  be an element of  $\mathcal{P}$  and let  $A$  have the spectral resolution  $\sum A_j E_j$ . Define the trace  $\text{Tr}(A)$  of  $A$  to be the least upper bound of the increasing sequence  $(\sum \{A_j \mid 1 \leq j \leq n\})_n$  in  $\mathcal{Z}$ . The trace of  $A$  is a function in  $\mathcal{P}$  since the elements  $A_j$  are uniquely determined by  $A$ . In fact we have the following proposition.

**PROPOSITION 4.3.** *Let  $A$  be an element of the set  $\mathcal{P}$ . Let  $S$  be a set of mutually orthogonal abelian projections in  $\mathcal{A}$  of least upper bound 1; let  $F(S)$  be the set of all finite subsets of  $S$ . Then*

$$\text{lub} \left\{ \sum \{ \tau_F(A) \mid F \in \pi \} \mid \pi \in F(S) \right\} = \text{Tr}(A).$$

**Proof.** From the proof of Proposition 4.2 we obtain that  $\sum \{ \tau_F(A) \mid F \in \pi \} \leq \text{Tr}(A)$  for all  $\pi \in F(S)$ . So  $B = \text{lub} \left\{ \sum \{ \tau_F(A) \mid F \in \pi \} \mid \pi \in F(S) \right\} \leq \text{Tr}(A)$ . Conversely, let  $n$  be a positive integer. For every  $\pi \in F(S)$  we have that

$$\sum \left\{ \tau_F \left( \sum \{ A_j E_j \mid 1 \leq j \leq n \} \right) \mid F \in \pi \right\} \leq B$$

since  $\sum \{ A_j E_j \mid 1 \leq j \leq n \} \leq A$ . So  $\sum \{ A_j \tau_{E_j}(\sum \{ F \mid F \in \pi \}) \mid 1 \leq j \leq n \} \leq B$  for every  $\pi \in F(S)$ . However, for every abelian projection  $E$ , we have that

$$\text{lub} \{ \tau_E(\sum \{ F \mid F \in \pi \}) \mid \pi \in F(S) \} = 1.$$

Indeed,  $\lim_\pi \sum \{ F \mid F \in \pi \} = 1$  (strongly) and therefore  $\lim_\pi \tau_E(\sum \{ F \mid F \in \pi \}) = 1$  (ultraweakly). Since  $\{ \tau_E(\sum \{ F \mid F \in \pi \}) \mid \pi \in F(S) \}$  is bounded above, the strong limit of this increasing net exists and must be equal to the ultraweak limit 1. This shows that

$$\text{lub}_\pi \sum \left\{ A_j \tau_{E_j} \left( \sum \{ F \mid F \in \pi \} \right) \mid 1 \leq j \leq n \right\} = \sum \{ A_j \mid 1 \leq j \leq n \}$$

is less than or equal to  $B$ . Because  $n$  is arbitrary  $\text{Tr}(A) \leq B$ . Q.E.D.

Now we show that  $\mathcal{P}$  is the set of positive elements of a two-sided ideal  $T$  in  $\mathcal{A}$ .

**THEOREM 4.4.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $I_a$  be the ideal generated by the abelian projections of  $\mathcal{A}$ . Let  $\mathcal{P}$  be the set of positive elements*

$A$  in  $I_a$  such that if  $\sum A_j E_j$  is a spectral representation of  $A$  then  $(\sum \{A_j \mid 1 \leq j \leq n\})_n$  is bounded above. Let  $\mathcal{P} = \{A \in \mathcal{A} \mid A^*A \in \mathcal{P}\}$ ; the set  $\mathcal{P}$  is a two-sided ideal in  $\mathcal{A}$ .

**Proof.** It is sufficient to show that the following three facts about  $\mathcal{P}$  are true: (1)  $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$ ; (2) if  $A \in \mathcal{A}$  and  $A^*A \in \mathcal{P}$ , then  $AA^* \in \mathcal{P}$ ; and (3) if  $A \in \mathcal{A}$ ,  $B \in \mathcal{P}$  and  $0 \leq A \leq B$ , then  $A \in \mathcal{P}$ .

First let  $A$  and  $B$  be elements of  $\mathcal{P}$ ; then  $A \in I_a^+$  and  $B \in I_a^+$ . Let  $S$  be a set of orthogonal abelian projections and let  $F(S)$  be the set of all finite subsets of  $S$ . We have that  $\sum \{\tau_F(A+B) \mid F \in \pi\} = \sum \{\tau_F(A) \mid F \in \pi\} + \sum \{\tau_F(B) \mid F \in \pi\} \leq \text{Tr}(A) + \text{Tr}(B)$ . So  $A+B \in \mathcal{P}$ .

Now let  $A \in \mathcal{A}$  and  $A^*A \in \mathcal{P}$ . Let  $\sum A_j E_j$  be a spectral resolution of  $A^*A$ . Let  $U$  be a partial isometric operator of  $\mathcal{A}$  such that  $U^*U$  is the domain support of  $A$ ,  $UU^*$  is the range support of  $A$  and  $U|A| = A$  ( $|A| = (A^*A)^{1/2}$ ). We have that  $\sum E_j$  is the domain support of  $A^*A$  and thus  $\sum E_j = U^*U$ . The projection  $F_j = UE_jU^*$  is abelian and equivalent to  $E_j$ . Also  $AA^* = U|A||A|U^* = U(\sum A_j E_j)U^* = \sum A_j F_j$ . However,  $\sum A_j F_j$  is a spectral resolution for  $AA^*$ ; thus,  $AA^* \in \mathcal{P}$ . Notice that  $\text{Tr}(A^*A) = \text{Tr}(AA^*)$ .

Finally, let  $A \in \mathcal{A}$ ,  $B \in \mathcal{P}$ , and  $0 \leq A \leq B$ . Let  $S$  be a set of mutually orthogonal abelian projections and let  $F(S)$  be the set of all finite subsets of  $S$ . We have that  $\sum \{\tau_F(A) \mid F \in \pi\} \leq \sum \{\tau_F(B) \mid F \in \pi\} \leq \text{Tr}(B)$  for every  $\pi \in F(S)$ . This proves  $A \in \mathcal{P}$ . Q.E.D.

We define the Schmidt class of a Type I von Neumann algebra  $\mathcal{A}$  to be the ideal  $\mathcal{S}$  and the trace class of the algebra  $\mathcal{A}$  to be the ideal  $\mathcal{T} = \mathcal{S}^2$ . The set of positive elements of  $\mathcal{T}$  is then precisely the set  $\mathcal{P}$ .

The next proposition extends the trace  $\text{Tr}$  to  $\mathcal{T}$ .

**PROPOSITION 4.5.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $\mathcal{T}$  be the trace class of  $\mathcal{A}$ . There is one and only one function  $\text{Tr}$  on  $\mathcal{T}$  into the center  $\mathcal{Z}$  of  $\mathcal{A}$  with the following properties: (1) if  $A$  is a positive element in  $\mathcal{T}$  with spectral resolution  $A = \sum A_j E_j$  then  $\text{Tr}(A) = \text{lub}_n \sum \{A_j \mid 1 \leq j \leq n\}$ ; (2) if  $A_1, A_2$  are elements of  $\mathcal{T}$  and  $C_1, C_2$  are elements of  $\mathcal{Z}$  then  $\text{Tr}(C_1 A_1 + C_2 A_2) = C_1 \text{Tr}(A_1) + C_2 \text{Tr}(A_2)$ ; (3) if  $A \in \mathcal{T}$ , then  $\text{Tr}(U^*AU) = \text{Tr}(A)$  for every unitary operator  $U$  in  $\mathcal{A}$ ; and (4) if  $A \in \mathcal{T}$  the function  $f(B) = \text{Tr}(BA)$  is continuous on  $\mathcal{A}$ .*

**Proof.** Let  $A_1$  and  $A_2$  be elements of  $\mathcal{P}$  and let  $S$  be a set of mutually orthogonal abelian projections of least upper bound 1; let  $F(S)$  be the set of all finite subsets of  $S$ . Suppose  $\sum A_{kj} E_{kj}$  is a spectral resolution for  $A_k$  ( $k=1, 2$ ). Let  $n$  be given. Then

$$\begin{aligned} & \sum \{A_{1j} + A_{2j} \mid 1 \leq j \leq n\} \\ &= \text{lub}_\pi \sum \left[ \tau_F \left( \sum \{A_{kj} E_{kj} \mid 1 \leq j \leq n; k=1, 2\} \right) \mid F \in \pi \right] \\ &\leq \text{lub}_\pi \sum \left[ \tau_F(A_1 + A_2) \mid F \in \pi \right] \leq \text{Tr}(A_1) + \text{Tr}(A_2). \end{aligned}$$

Therefore,  $\text{Tr}(A_1) + \text{Tr}(A_2) \leq \text{Tr}(A_1 + A_2) \leq \text{Tr}(A_1) + \text{Tr}(A_2)$ . This proves that  $\text{Tr}(A_1 + A_2) = \text{Tr}(A_1) + \text{Tr}(A_2)$  if  $A_1, A_2$  are in  $\mathcal{P}$ .

Now let  $C \in \mathcal{Z}^+$  and let  $A \in \mathcal{P}$ . Assume  $C \neq 0$ . Let  $P$  be the projection in  $\mathcal{Z}$  corresponding to the closure of the open set of all  $\zeta$  in the spectrum of  $\mathcal{Z}$  such that  $C^\wedge(\zeta) \neq 0$ . Then  $CA$  has the spectral resolution  $\sum (CA_j)(E_j P)$  where  $\sum A_j E_j$  is a spectral resolution of  $A$ . We have that

$$\text{Tr}(CA) = \text{lub}_n \sum \{CA_j \mid 1 \leq j \leq n\} = C \text{lub}_n \sum \{A_j \mid 1 \leq j \leq n\} = C \text{Tr}(A).$$

If  $A$  is a self-adjoint element in the trace class, then  $A$  may be written as the difference  $A = A_1 - A_2$  of two elements  $A_1$  and  $A_2$  in  $\mathcal{P}$ . Setting  $\text{Tr}(A) = \text{Tr}(A_1) - \text{Tr}(A_2)$  uniquely defines the function  $\text{Tr}$  on the self-adjoint elements of the trace class. If  $A$  is an arbitrary element of the trace class, then  $A$  may be written as a linear combination  $A = A_1 + iA_2$  of two self-adjoint elements  $A_1$  and  $A_2$  of the trace class. Setting  $\text{Tr}(A) = \text{Tr}(A_1) + i \text{Tr}(A_2)$  uniquely defines  $\text{Tr}$  on elements of the trace class. The function  $\text{Tr}$  will then have properties (1) and (2).

Now let  $A \in \mathcal{P}$  and let  $A = \sum A_j E_j$  be a resolution of  $A$ ; let  $U$  be a unitary element of  $\mathcal{A}$ . We see that  $U^*AU \in \mathcal{P}$  and  $\sum A_j U^* E_j U$  is a resolution for  $U^*AU$ . Thus,  $\text{Tr}(A) = \text{Tr}(U^*AU)$ . Since every  $A$  in  $\mathcal{T}$  is a linear combination of four elements in  $\mathcal{P}$ ,  $\text{Tr}(U^*AU) = \text{Tr}(A)$  for every  $A$  in  $\mathcal{P}$ .

Finally, let  $A \in \mathcal{T}^+$  and let  $B \in \mathcal{A}^+$ . We have  $\text{Tr}(A^{1/2}BA^{1/2}) \leq \|B\| \text{Tr}(A)$  since  $0 \leq A^{1/2}BA^{1/2} \leq \|B\|A$ . So  $\text{Tr}(BA) \leq \|B\| \text{Tr}(A)$ . If  $B$  is an arbitrary element of  $\mathcal{A}$ , we may write  $B = B_1 - B_2 + i(B_3 - B_4)$  where  $B_j$  is positive and  $B_1 B_2 = B_3 B_4 = 0$ . Thus  $\|\text{Tr}(BA)\| \leq \sum (\|B_j\| \|\text{Tr}(A)\|) \leq 4\|B\| \|\text{Tr}(A)\|$ . If  $A$  is an arbitrary element of  $\mathcal{T}$ , we may write  $A = A_1 - A_2 + i(A_3 - A_4)$  where the  $A_j$  are elements of  $\mathcal{T}^+$ . So  $\|\text{Tr}(BA)\| \leq 4\|B\| \sum \|\text{Tr}(A_j)\|$ . This shows that the function  $B \rightarrow \text{Tr}(BA)$  is continuous on  $\mathcal{A}$  for fixed  $A$  in  $\mathcal{T}$ . Q.E.D.

Let  $A$  and  $B$  be two elements of the Schmidt class. Define the function of  $\mathcal{S} \times \mathcal{S}$  into  $\mathcal{Z}$  by  $(A, B) = \text{Tr}(B^*A)$ . Then this function satisfies the properties: (1)  $(A, B) = (B, A)^*$ ; (2)  $(A, A) \geq 0$  and  $(A, A) = 0$  if and only if  $A = 0$ ; and (3)  $(CA + C'A', B) = C(A, B) + C'(A', B)$  for  $C, C' \in \mathcal{Z}$ ,  $A' \in \mathcal{S}$ . Define  $|A| = (A, A)^{1/2}$  and  $\|A\|_1 = \|(A, A)\|^{1/2}$ . We have that  $\|A\| \leq \|A\|_1$  for all  $A$  in  $\mathcal{S}$ . Indeed, let  $\sum A_j E_j$  be a spectral resolution for  $A^*A$ . We have that  $\|A_1\| = \|A^*A\|$ . So  $\|A\|^2 = \|A^*A\| = \|A_1\| \leq \|\text{Tr}(A^*A)\| = \|A\|_1^2$ . The function  $A \rightarrow \|A\|_1$  on  $\mathcal{S}$  defines a norm [9]. We show that  $\mathcal{S}$  is complete in this norm.

**THEOREM 4.6.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $\mathcal{S}$  be the Schmidt class of  $\mathcal{A}$ . Under the norm  $\|A\|_1^2 = \|\text{Tr}(A^*A)\|$ , the Schmidt class  $\mathcal{S}$  is complete.*

**Proof.** Let  $\{A_n\}$  be a Cauchy sequence in  $\mathcal{S}$ . The relation  $\|B\| \leq \|B\|_1$  for  $B \in \mathcal{S}$  implies that  $\{A_n\}$  is a Cauchy sequence in  $I_a$ . There is an element  $A$  in  $I_a$  such that  $\lim_n \|A_n - A\| = 0$ . We prove that  $A \in \mathcal{S}$  and that  $\lim_n \|A_n - A\|_1 = 0$ .

Let  $S$  be a set of mutually orthogonal abelian projections and let  $F(S)$  be the set of all finite subsets of  $S$ . For each  $\pi \in F(S)$  we have that

$$\begin{aligned} & \left[ \sum \{ \tau_F((A - A_n)^*(A - A_n)) \mid F \in \pi \} \right]^{1/2} \\ & \leq \left[ \sum \{ \tau_F((A - A_m)^*(A - A_m)) \mid F \in \pi \} \right]^{1/2} \\ & \quad + \left[ \sum \{ \tau_F((A_m - A_n)^*(A_m - A_n)) \mid F \in \pi \} \right]^{1/2}. \end{aligned}$$

Indeed, let  $\zeta$  be an arbitrary point in the spectrum of the center of  $\mathcal{A}$ . For each  $F \in \pi$ ,  $F(\zeta)$  is a projection on  $H(\zeta)$  of dimension at most one. Let  $x_1, x_2, \dots, x_n$  be an orthonormal set of vectors in  $H(\zeta)$  which respectively span the subspaces corresponding to the nonzero  $F(\zeta)$  with  $F \in \pi$ . Then

$$\begin{aligned} & \left[ \sum \{ \tau_F((A - A_n)^*(A_n - A)) \mid F \in \pi \} \right]^{1/2 \wedge (\zeta)} \\ & = \left[ \sum \| (A(\zeta) - A_n(\zeta))x_j \|^2 \right]^{1/2} \\ & \leq \left[ \sum \| (A(\zeta) - A_m(\zeta))x_j \|^2 \right]^{1/2} + \left[ \sum \| (A_m(\zeta) - A_n(\zeta))x_j \|^2 \right]^{1/2} \\ & = \left[ \sum \{ \tau_F((A - A_m)^*(A - A_m)) \mid F \in \pi \} \right]^{1/2 \wedge (\zeta)} \\ & \quad + \left[ \sum \{ \tau_F((A_m - A_n)^*(A_m - A_n)) \mid F \in \pi \} \right]^{1/2 \wedge (\zeta)}. \end{aligned}$$

Since  $\zeta$  is arbitrary, we have the desired result. Therefore,

$$\begin{aligned} \left[ \sum \{ \tau_F((A - A_n)^*(A - A_n)) \mid F \in \pi \} \right]^{1/2} & \leq \left[ \sum \{ \tau_F((A - A_m)^*(A - A_m)) \mid F \in \pi \} \right]^{1/2} \\ & \quad + (A_n - A_m, A_n - A_m)^{1/2}. \end{aligned}$$

Since  $\lim_m \| \tau_F((A - A_m)^*(A - A_m)) \| = 0$  and since  $\{A_n\}$  is Cauchy, given any  $\varepsilon > 0$  there is an  $N$  such that  $n \geq N$  implies  $\| [\sum \{ \tau_F((A - A_n)^*(A - A_n)) \mid F \in \pi \}]^{1/2} \| < \varepsilon$ , for every  $\pi \in F(S)$ . This shows that  $A - A_n \in \mathcal{S}$  and thus that  $A \in \mathcal{S}$ . It also shows that  $\|A - A_n\|_1 \leq \varepsilon$  whenever  $n \geq N$ . Thus  $\lim_n \|A - A_n\|_1 = 0$ . We obtain that  $\mathcal{S}$  is complete. Q.E.D.

Let  $M$  be a module over a commutative  $AW^*$ -algebra  $\mathcal{Z}$ . Suppose there is an inner product  $(A, B)$  of  $M$  with values in  $\mathcal{Z}$  satisfying (1)  $(A, B) = (B, A)^*$ ; (2)  $(CA, B) = C(A, B)$ ; (3)  $(A + A', B) = (A, B) + (A', B)$ ; and (4)  $(A, A) \geq 0$  and  $(A, A) = 0$  if and only if  $A = 0$ , for all  $A, A', B$  in  $M$  and  $C$  in  $\mathcal{Z}$ . By setting  $\|A\|_1 = \|(A, A)^{1/2}\|$  for each  $A$  in  $M$  we define a norm in  $M$ . The  $\mathcal{Z}$ -module  $M$  is said to be an  $AW^*$ -module over  $\mathcal{Z}$  if (1)  $M$  is complete; (2) if  $R$  is a set of mutually orthogonal projections in  $\mathcal{Z}$  of least upper bound 1 and if  $A \in M$ , then  $PA = 0$  for every  $P \in R$  implies  $A = 0$ ; and (3) if  $R = \{P_j \mid j \in J\}$  is a set of mutually orthogonal projections in  $\mathcal{Z}$  of least upper bound 1 and if  $\{A_j \mid j \in J\}$  is a bounded set of elements of  $M$ , then there is an  $A \in M$  such that  $P_j A = P_j A_j$  for every  $j \in J$  [9]. We show that the Schmidt class of a Type I von Neumann algebra  $\mathcal{A}$  is an  $AW^*$ -module over the center.

**THEOREM 4.7.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$ . Let  $\mathcal{S}$  be the Schmidt class of  $\mathcal{A}$ . Under the inner product  $\text{Tr}(B^*A) = (A, B)$  on  $\mathcal{S}$ , the ideal  $\mathcal{S}$  is an  $AW^*$ -module over  $\mathcal{Z}$ .*

**Proof.** The only point which remains to be verified is the following: if  $\{P_j \mid j \in J\}$  is a set of mutually orthogonal central projections of least upper bound 1 and if  $\{A_j \mid j \in J\}$  is a bounded set of elements in  $\mathcal{S}$ , there is an  $A \in \mathcal{S}$  such that  $AP_j = A_jP_j$  for all  $j$ . However, let  $\|\text{Tr}(A_j^*A_j)\| \leq n$  for all  $j$ ; then  $\|A_j\|^2 \leq \|\text{Tr}(A_j^*A_j)\| \leq n$ . Let  $A = \sum A_jP_j$  in  $\mathcal{A}$ . Let  $S$  be a set of mutually orthogonal abelian projections. For each finite subset  $\pi$  of  $S$  and for each  $j \in J$  we have

$$P_j\left(\sum \{\tau_F(A^*A) \mid F \in \pi\}\right) = P_j\left(\sum \{\tau_F(A_j^*A_j) \mid F \in \pi\}\right) \leq \text{Tr}(A_j^*A_j)P_j \leq nP_j.$$

Thus,

$$\sum \{\tau_F(A^*A) \mid F \in \pi\} \leq n \cdot 1.$$

However, the set  $\pi$  is arbitrary and  $\{\sum \{\tau_F(A^*A) \mid F \in \pi\} \mid \pi \text{ a finite subset of } S\}$  is bounded above. Therefore  $A \in \mathcal{S}$ . So  $\mathcal{S}$  is an  $AW^*$ -module over  $\mathcal{Z}$ .

Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $M$  be an  $AW^*$ -module over  $\mathcal{Z}$ . A function  $\phi$  of  $I_a$  (respectively,  $M$ ) into  $\mathcal{Z}$  is called a  $\mathcal{Z}$ -linear functional if  $\phi(C_1A_1 + C_2A_2) = C_1\phi(A_1) + C_2\phi(A_2)$  for all  $C_1, C_2 \in \mathcal{Z}$  and  $A_1, A_2 \in I_a$  (respectively,  $A_1, A_2 \in M$ ). We now show that the set of all uniformly continuous  $\mathcal{Z}$ -linear functionals on  $I_a$  is identifiable with the trace class of  $\mathcal{A}$ . We need the following lemma to show this.

**LEMMA.** *Let  $M$  be an  $AW^*$ -module over the  $AW^*$ -module  $\mathcal{Z}$ . If  $\phi$  is a continuous  $\mathcal{Z}$ -linear functional on  $M$ , there is a unique  $A_0 \in M$  such that  $\phi(A) = (A, A_0)$  for all  $A \in M$  [9, Theorem 5].*

**THEOREM 4.8.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra on the Hilbert space  $H$  and let  $\mathcal{Z}$  be the center of  $\mathcal{A}$ . Let  $I_a$  be the ideal generated by the abelian projections of  $\mathcal{A}$ , let  $\mathcal{T}$  be the trace class of  $\mathcal{A}$  and let  $\text{Tr}$  denote the trace function on  $\mathcal{T}$ . If  $\phi$  is a uniformly continuous  $\mathcal{Z}$ -linear functional of  $I_a$  into  $\mathcal{Z}$ , there is a unique  $A_0$  in  $\mathcal{T}$  such that  $\phi(A) = \text{Tr}(AA_0)$  for all  $A \in I_a$ .*

**Proof.** We first settle the question of the uniqueness of  $A_0$ . Suppose  $\text{Tr}(AA_0) = \text{Tr}(AA'_0)$  for all  $A \in I_a$ . In particular if we let  $A = (A_0 - A'_0)^*$ , then  $\|A_0 - A'_0\|_1 = 0$ . So  $A_0 = A'_0$ .

The uniformly continuous  $\mathcal{Z}$ -linear functional  $\phi$  is a linear combination of two uniformly continuous  $\mathcal{Z}$ -linear functionals  $\phi_1(A) = \phi(A) + \phi(A^*)^*$  and  $\phi_2(A) = i(\phi(A) - \phi(A^*)^*)$ . For each  $A \in I_a$ , we have that  $\phi_j(A) = \phi_j(A^*)^*$  ( $j = 1, 2$ ). So we may assume that  $\phi(A) = \phi(A^*)^*$  for each  $A \in I_a$ .

Let  $\psi$  be the restriction of  $\phi$  to the Schmidt class  $\mathcal{S}$ . Because  $\|A\| \leq \|A\|_1$  for each  $A \in \mathcal{S}$ ,  $\psi$  is a  $\mathcal{Z}$ -linear functional on  $\mathcal{S}$  which is continuous in the topology on  $\mathcal{S}$  induced by  $\|\cdot\|_1$ . By the lemma, there is a unique  $A_0 \in \mathcal{S}$  such that  $\psi(A) = \text{Tr}(A_0A)$



for each  $A \in \mathcal{S}$ . For each abelian projection  $E$  in  $\mathcal{A}$ ,  $\psi(E) = \psi(E^*)^* = \psi(E)^*$ ; therefore  $\tau_E(A_0) \operatorname{Tr}(E) = \psi(E)$  is self-adjoint. Thus,  $A_0$  is self-adjoint.

We prove that  $A_0$  is an element of the trace class  $\mathcal{T}$ . In fact let  $A_1$  and  $A_2$  be the positive and negative parts of  $A_0$  respectively. Let  $P$  be the domain support of  $A_1$  and let  $S$  be a set of mutually orthogonal abelian projections majorized by  $P$ . Then for any finite subset  $\pi$  of  $S$  we have that

$$\left\| \sum \{ \tau_F(A_0) \mid F \in \pi \} \right\| = \left\| \operatorname{Tr} \left( \sum \{ F \mid F \in \pi \} A_0 \right) \right\| = \left\| \phi \left( \sum \{ F \mid F \in \pi \} \right) \right\| \leq \|\phi\|.$$

So  $\left\{ \sum \{ \tau_F(A_0) \mid F \in \pi \} \right\}_\pi$  is bounded above. This proves that  $A_1 = A_0 P$  is an element of  $\mathcal{T}$ . Similarly  $A_2 \in \mathcal{T}$ .

Finally the function  $A \rightarrow \operatorname{Tr}(AA_0)$  is continuous on  $\mathcal{A}$ . The relation  $\operatorname{Tr}(EA_0) = \phi(E)$  for every abelian projection  $E$  in  $I_a$  implies that  $\phi(A) = \operatorname{Tr}(AA_0)$  for all  $A$  in  $I_a$ . Q.E.D.

Let  $L(I_a, \mathcal{Z})$  be the Banach space of all bounded linear transformations of  $I_a$  into  $\mathcal{Z}$ . Let  $L = L_{\mathcal{Z}}(I_a, \mathcal{Z})$  be the set of all  $\mathcal{Z}$ -linear transformations in  $L(I_a, \mathcal{Z})$ . The set  $L$  is a linear manifold and is closed in the norm topology of  $L(I_a, \mathcal{Z})$ . In fact if  $\{\phi_n\}$  is a sequence in  $L(I_a, \mathcal{Z})$  which converges to  $\phi$ , then for each  $C \in \mathcal{Z}$  and  $A \in I_a$  we have  $\phi(CA) = \lim_n \phi_n(CA) = C \lim_n \phi_n(A) = C\phi(A)$ . This shows that  $L$  is a Banach space under the norm induced on  $L$  by  $L(I_a, \mathcal{Z})$ .

**THEOREM 4.9.** *Let  $\mathcal{T}$  be the trace class of the Type I von Neumann algebra  $\mathcal{A}$  with center  $\mathcal{Z}$ . For each  $A \in \mathcal{T}$  let  $\|A\|_2 = \|\operatorname{Tr}((A^*A)^{1/2})\|$ ; the function  $\|\cdot\|_2$  is a norm on  $\mathcal{T}$  under which  $\mathcal{T}$  is a Banach algebra with involution. If  $L = L_{\mathcal{Z}}(I_a, \mathcal{Z})$  is the Banach space of all continuous  $\mathcal{Z}$ -linear functionals on  $I_a$  and if  $A_\phi$  is the unique element in  $\mathcal{T}$  such that  $\phi(A) = \operatorname{Tr}(AA_\phi)$  where  $\phi \in L$ , then the function  $\phi \rightarrow A_\phi$  is an isometric isomorphism of  $L$  onto  $\mathcal{T}$  (as a Banach space).*

**Proof.** Because the function  $\phi \rightarrow A_\phi$  is an isomorphism, it is sufficient to prove  $\|\phi\| = \|\operatorname{Tr}((A_\phi^*A_\phi)^{1/2})\|$  in order to show that  $\mathcal{T}$  is a Banach space and that the function  $\phi \rightarrow \|A_\phi\|$  is an isometry. Let  $A' = A_\phi$  and  $|A'| = (A'^*A')^{1/2}$ . Let  $U$  be a partial isometric operator in  $\mathcal{A}$  such that  $U|A'| = A'$ . For each  $A \in I_a$  we have

$$\begin{aligned} \|\phi(A)\| &= \|\operatorname{Tr}(AA')\| \leq \|\operatorname{Tr}((AU|A'|^{1/2})(AU|A'|^{1/2}))^{1/2} \operatorname{Tr}(|A'|)\|^{1/2} \\ &\leq \|U^*A^*AU\|^{1/2} \|\operatorname{Tr}(|A'|)\| \leq \|A\| \|\operatorname{Tr}(|A'|)\|. \end{aligned}$$

Therefore,  $\|\phi\| \leq \|\operatorname{Tr}(|A'|)\|$ . Conversely, let  $\sum A_j E_j$  be a spectral resolution for  $|A'|$ . We have  $\operatorname{lub}_n \|\sum \{A_j \mid 1 \leq j \leq n\}\| = \|\operatorname{Tr}(|A'|)\|$ . Because  $F_n = \sum \{E_j \mid 1 \leq j \leq n\}$  is an element of  $I_a$  of norm 1, we obtain  $\|\phi\| \geq \|\operatorname{Tr}(F_n U^* A')\| = \|\operatorname{Tr}(F_n |A'|)\| = \|\sum \{A_j \mid 1 \leq j \leq n\}\|$ . Hence  $\|\phi\| \geq \|\operatorname{Tr}(|A'|)\|$ . This shows  $\|\phi\| = \|\operatorname{Tr}(|A'|)\|$ .

Because  $\|A\| = \||A|\| \leq \|\operatorname{Tr}(|A|)\|$  for every  $A \in \mathcal{T}$ , we have  $\|AB\|_2 \leq \|A\| \|B\|_2 \leq \|A\|_2 \|B\|_2$  for every  $A$  and  $B$  in  $\mathcal{T}$ . So  $\mathcal{T}$  is a Banach algebra.

Finally, let  $A \in \mathcal{T}$  and let  $U|A| = A$  be the polar decomposition of  $A$ . Then  $|A^*| = U|A|U^*$ . Therefore,  $\|A^*\|_2 = \operatorname{Tr}(U|A|U^*) = \operatorname{Tr}(|A|) = \|A\|_2$ . Therefore,  $\mathcal{T}$  is a Banach \*-algebra. Q.E.D.

**THEOREM 4.10.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  over the Hilbert space  $H$  and let  $\mathcal{T}$  be the trace class considered as a Banach space with that topology induced by the norm  $\|\cdot\|_2$ . Let  $A_0 \in \mathcal{A}$ ; the function  $\phi(A) = \text{Tr}(AA_0)$  is a continuous  $\mathcal{Z}$ -linear function of  $\mathcal{T}$  to  $\mathcal{Z}$ . Conversely, if  $\phi$  is a continuous  $\mathcal{Z}$ -linear function of  $\mathcal{T}$  into  $\mathcal{Z}$ , there is a unique  $A_\phi \in \mathcal{A}$  such that  $\phi(A) = \text{Tr}(AA_\phi)$  for all  $A \in \mathcal{T}$ . The function  $\phi \rightarrow A_\phi$  is an isometric isomorphism of the Banach space of all continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{T}$  into  $\mathcal{Z}$  onto  $\mathcal{A}$ .*

**Proof.** Let  $A_0 \in \mathcal{A}$  and let  $A \in \mathcal{T}$ . Then we have

$$\begin{aligned} \|\text{Tr}(AA_0)\| &= \|\text{Tr}(U|A|A_0)\| = \|\text{Tr}(|A|^{1/2}|A|^{1/2}A_0U)\| \\ &\leq \|\text{Tr}(|A|)^{1/2}\| \|\text{Tr}(U^*A_0^*|A|A_0U)^{1/2}\| \\ &\leq \|\text{Tr}(|A|)^{1/2}\|^2 \|A_0UU^*A_0^*\|^{1/2} \leq \|A_0\| \|\text{Tr}(|A|)\|, \end{aligned}$$

where  $U|A|$  is the polar decomposition of  $A$ . Thus  $\phi(A) = \text{Tr}(AA_0)$  is a continuous  $\mathcal{Z}$ -linear function on  $\mathcal{T}$  and  $\|\phi\| \leq \|A_0\|$ . On the other hand, given  $\varepsilon > 0$  there are two abelian projections  $E$  and  $F$  both having central support 1 such that  $\|EA_0F\| \geq \|A_0\| - \varepsilon$ . There is a partial isometric operator  $U$  in  $\mathcal{T}$  such that  $U^*U = E$  and  $UU^* = F$ . Because  $\text{Tr}(|U|) = \text{Tr}(E) = 1$ , we have that

$$\begin{aligned} \|\phi\| &\geq \|\text{Tr}(UA_0)\| = \|\text{Tr}(\tau_F(UA_0)F)\| \\ &= \|\tau_F(UA_0)\| = \|FA_0^*U^*UA_0F\|^{1/2} = \|EA_0F\| \geq \|A_0\| - \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we have  $\|\phi\| \geq \|A_0\|$ . Hence, we have proved that  $\|\phi\| = \|A_0\|$ .

Let  $\phi$  be a continuous  $\mathcal{Z}$ -linear functional of  $\mathcal{T}$  into  $\mathcal{Z}$ . Let  $\mathcal{S}$  be the Schmidt class of  $\mathcal{A}$  with the norm  $\|A\|_1 = \|\text{Tr}(A^*A)^{1/2}\|$ . For every fixed  $B \in \mathcal{S}$  the function  $A \rightarrow \phi(B^*A)$  on  $\mathcal{S}$  is a continuous  $\mathcal{Z}$ -linear functional. In fact,  $\|\phi(B^*A)\| \leq \|\phi\| \|\text{Tr}(|B^*A|)\|$ . Now we obtain an estimate for  $\|\text{Tr}(|B^*A|)\|$  [12, Lemma 5.14]. Let  $A = U|A|$  be the polar decomposition of  $A$  and let  $B^*A = V|B^*A|$  be the polar decomposition of  $B^*A$ . Then  $|B^*A| = V^*B^*A = V^*B^*U|A|$ . Because  $V^*B^*U$  and  $|A|$  are elements of  $\mathcal{S}$  we have

$$\begin{aligned} \|\text{Tr}(|B^*A|)\| &\leq \|\text{Tr}((V^*B^*U)^*(V^*B^*U))^{1/2}\| \cdot \|\text{Tr}(|A|^2)^{1/2}\| \\ &= \|\text{Tr}(U^*BV^*VB^*U)^{1/2}\| \|\text{Tr}(|A|^2)^{1/2}\| \\ &\leq \|U\| \|V\| \|\text{Tr}(B^*B)^{1/2}\| \|\text{Tr}(A^*A)^{1/2}\| \leq \|B\|_1 \|A\|_1. \end{aligned}$$

Therefore  $\|\phi(B^*A)\| \leq \|\phi\| \|B\|_1 \|A\|_1$ .

There is for every  $B \in \mathcal{S}$  a unique  $\Phi(B) \in \mathcal{S}$  such that  $\phi(B^*A) = \text{Tr}(\Phi(B)^*A) = (A, \Phi(B))$  for every  $A \in \mathcal{S}$ . Let  $F$  be a projection in  $I_a$ . Define  $\theta(F) = F\Phi(F)F$ . Let  $E$  be an abelian projection. Then

$$\begin{aligned} \|\tau_E(\Phi(F))\| &= \|(E, \Phi(F))\| = \|\phi(FE)\| \leq \|\phi\| \|\text{Tr}(|FE|)\| \\ &= \|\phi\| \|\text{Tr}((EFFE)^{1/2})\| = \|\phi\| \|\text{Tr}((\tau_E(F))^{1/2}E)\| \\ &= \|\phi\| \|\tau_E(F)^{1/2}\| \leq \|\phi\|. \end{aligned}$$

Therefore,  $\|\Phi(F)\| \leq 2\|\phi\|$  and  $\|\theta(F)\| \leq 2\|\phi\|$ .

Let  $S$  be the set of all projections in  $I_a$ ; the least upper bound (in the lattice of projections of  $\mathcal{A}$ ) of two projections of  $I_a$  is again a projection in  $I_a$ . Therefore,  $S$  is a directed set under the usual ordering for projections.

If  $F$  and  $G$  are two projections in  $S$  such that  $F \leq G$ , then  $\theta(F) = F\theta(G)F$ . Indeed, for every  $A \in \mathcal{S}$  we have  $(A, \theta(F)) = (FAF, \Phi(F)) = \phi(FAF) = \phi(GFAFG) = (GFAFG, \Phi(G)) = (A, F\theta(G)F)$ . Hence, if  $x \in H$  and if there is a projection  $F \in S$  such that  $Fx = x$ , then the net  $\{(\theta(E)x, x) \mid E \in S\}$  is eventually constant. So for every such  $x \in H$ ,  $\lim_S (\theta(E)x, x)$  exists. Let  $x$  be an arbitrary element in  $H$ . We show that  $\lim_S (\theta(E)x, x)$  exists by showing that the net  $\{(\theta(E)x, x) \mid E \in S\}$  is Cauchy. Let  $\varepsilon > 0$  be given. Let  $G$  be an element in  $S$  such that  $\|(1-G)x\| < \varepsilon$  and let  $y = Gx$ . For every  $F$  and  $F'$  in  $S$  such that  $F \geq G$  and  $F' \geq G$  we have  $(\theta(F)y, y) = (\theta(F')y, y)$ . Thus,

$$\begin{aligned} |(\theta(F)x, x) - (\theta(F')x, x)| &\leq |(\theta(F)x, x) - (\theta(F)y, y)| \\ &\quad + |(\theta(F')y, y) - (\theta(F')x, x)| \leq |(\theta(F)(x-y), x)| \\ &\quad + |(\theta(F)y, x-y)| + |(\theta(F')(y-x), y)| \\ &\quad + |(\theta(F')x, y-x)| \leq 8\varepsilon\|\phi\| \|x\| \end{aligned}$$

because  $\|\theta(E)\| \leq 2\|\phi\|$  for all  $E \in S$ . This shows that for every  $x \in H$  the net  $\{(\theta(E)x, x) \mid E \in S\}$  converges. By the standard arguments there is a bounded linear operator  $A_0$  on  $H$  such that (1)  $\|A_0\| \leq 2\|\phi\|$  and (2)  $\lim_S (\theta(E)x, y) = (A_0x, y)$  for every  $x, y \in H$ . Therefore,  $A_0$  is an element of  $\mathcal{A}$ .

Now let  $E$  be a projection in the set  $S$ . We have  $(EA_0Ex, y) = \lim_S (\theta(F)Ex, Ey) = (\theta(E)x, y)$ . This means that  $EA_0E = \theta(E)$ .

Let  $B \in \mathcal{S}$  and  $C \in \mathcal{S}$  and let  $E$  be a projection in  $I_a$ . We have that the range projection of  $B^*ECE$  is equivalent to a projection majorized by  $E$ . So the range projection of  $B^*ECE$  is in the set  $S$ . Let  $F$  be a projection in  $S$  majorizing both  $E$  and the range projection of  $B^*ECE$ . Then

$$\begin{aligned} (C, EBA_0E) &= (ECE, BA_0) = (B^*ECE, A_0E) = (FB^*ECE, A_0E) \\ &= (B^*ECE, FA_0F) = (B^*ECE, \theta(F)) = (B^*ECE, \Phi(F)) \\ &= \phi(FB^*ECE) = \phi(B^*ECE) = (ECE, \Phi(B)) = (C, E\Phi(B)E). \end{aligned}$$

So for every  $E \in S$ ,  $E\Phi(B)E = EBA_0E$ . This proves that  $\Phi(B) = BA_0$ .

Finally, let  $B \in \mathcal{P}$ . Then  $B^{1/2} \in \mathcal{S}$ . We have

$$\phi(B) = \phi(B^{1/2}B^{1/2}) = (B^{1/2}, \Phi(B^{1/2})) = (B^{1/2}, B^{1/2}A_0) = \text{Tr}(A_0^*B).$$

Therefore, for every  $B \in \mathcal{T}$ ,  $\phi(B) = \text{Tr}(A_0^*B)$ .

If  $\text{Tr}(BA_0) = \text{Tr}(BA'_0)$  for every  $B \in \mathcal{T}$ , then  $\tau_E(A_0 - A'_0)$  for every abelian projection. This shows that  $A_0 = A'_0$ . Therefore, if  $\phi$  is a continuous  $\mathcal{Z}$ -linear function of function  $\mathcal{T}$  into  $\mathcal{Z}$  there is one and only one  $A_\phi$  in  $\mathcal{A}$  such that  $\phi(A) = \text{Tr}(AA_\phi)$  for all  $A \in \mathcal{T}$ . Since the function  $\phi \rightarrow A_\phi$  is linear (in fact it is  $\mathcal{Z}$ -linear),

the function is an isometric isomorphism of the Banach space of all continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{T}$  into  $\mathcal{Z}$  onto  $\mathcal{A}$ . Q.E.D.

We now study the properties of the trace  $\text{Tr}$  relative to the weak topology. If  $\mathcal{A}$  and  $\mathcal{B}$  are two von Neumann algebras, a linear function  $\Phi$  of  $\mathcal{A}$  into  $\mathcal{B}$  is said to be normal if (1)  $\Phi(\mathcal{A}^+) \subset \mathcal{B}^+$  and (2) if  $\{A_n\}$  is a monotonically increasing net in  $\mathcal{A}^+$  such that  $\text{lub}_n A_n = A (A \in \mathcal{A}^+)$  then  $\text{lub}_n \Phi(A_n) = \Phi(A)$ . A linear function  $\Phi$  of  $\mathcal{A}$  into  $\mathcal{B}$  which carries  $\mathcal{A}^+$  into  $\mathcal{B}^+$  is normal if and only if  $\Phi$  is continuous in the ultraweak topologies of  $\mathcal{A}$  and  $\mathcal{B}$ . The ultraweak topology for  $\mathcal{A}$  on the Hilbert space  $H$  is the weakest topology such that all linear functionals of the form  $A \rightarrow \sum \{(Ax_j, y_j) \mid 1 \leq j < \infty\}$  are continuous where  $\{x_j\}$  and  $\{y_j\}$  are sequences of  $H$  such that  $\sum \|x_j\|^2 < +\infty$  and  $\sum \|y_j\|^2 < +\infty$ .

If  $A$  is an element of the trace class the function  $B \rightarrow \text{Tr}(BA)$  is an ultraweakly continuous linear function of  $\mathcal{A}$  into its center. We prove this in the following form.

**THEOREM 4.11.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $\mathcal{T}$  be the trace class of  $\mathcal{A}$ . Let  $\mathcal{P}$  be the set of positive elements of  $\mathcal{T}$ . Assume that  $\{A_n \mid n \in D\}$  is a monotonically increasing net of elements of  $\mathcal{P}$  which is bounded above and suppose  $A = \text{lub}_n A_n$  is an element of  $\mathcal{P}$ . Then  $\text{lub}_n \text{Tr}(A_n) = \text{Tr}(A)$ .*

**Proof.** If  $x$  is an arbitrary vector of the Hilbert space of  $\mathcal{A}$ , it is sufficient to show  $\lim_n \text{Tr}(A_n)x = \text{Tr}(A)x$  in order to show  $\text{lub}_n \text{Tr}(A_n) = \text{Tr}(A)$  because  $\{\text{Tr}(A_n) \mid n \in D\}$  is a monotonically increasing net in  $\mathcal{Z}$  bounded above by  $\text{Tr}(A)$ . Let  $S$  be a set of mutually orthogonal abelian projections of least upper bound 1 and let  $F(S)$  be the collection of finite subsets of  $S$ . For each  $E \in S$  we have  $\tau_E(A_n) \leq \tau_E(A)$  and  $\lim_n \tau_E(A_n) = \tau_E(A)$  (strongly). Now given  $\varepsilon > 0$  there is an element  $\pi$  in  $F(S)$  such that  $\|\sum \{\tau_F(A) \mid F \in \pi'\}x\| < \varepsilon$  for every  $\pi' \in F(S)$  such that  $\pi' \cap \pi = \emptyset$ . So  $\|\sum \{\tau_F(A_n) \mid F \in \pi'\}x\| < \varepsilon$  for every  $\pi' \in F(S)$  such that  $\pi' \cap \pi = \emptyset$  and for every  $n \in D$ . There is an  $n_0 \in D$  such that  $\|(\sum \{\tau_F(A_n) \mid F \in \pi\} - \sum \{\tau_F(A) \mid F \in \pi\})x\| < \varepsilon$  whenever  $n \geq n_0$ . Therefore, if  $\pi'$  is in  $F(S)$  and  $\pi' \supset \pi$  we have

$$\|(\sum \{\tau_F(A_n) \mid F \in \pi'\} - \sum \{\tau_F(A) \mid F \in \pi'\})x\| < 3\varepsilon,$$

whenever  $n \geq n_0$ . Thus,  $\|(\text{Tr}(A_n) - \text{Tr}(A))x\| \leq 3\varepsilon$  whenever  $n \geq n_0$ . This proves that  $\lim_n \text{Tr}(A_n)x = \text{Tr}(A)x$  and that  $\lim_n \text{Tr}(A_n) = \text{Tr}(A)$  (strongly). Q.E.D.

**COROLLARY.** *Let  $A$  be an element of the trace class. The linear function  $\phi(B) = \text{Tr}(BA)$  of  $\mathcal{A}$  into  $\mathcal{Z}$  is ultraweakly continuous.*

**Proof.** We may write  $A$  as a linear combination of four positive elements in the trace class. So we may assume that  $A$  is an element of  $\mathcal{P}$ . If  $B \in \mathcal{A}^+$  then  $\phi(B) = \text{Tr}(BA) = \text{Tr}(B^{1/2}AB^{1/2}) \geq 0$ ; therefore  $\phi$  is a positive function. If  $\{B_n \mid n \in D\}$  is a monotonically increasing net in  $\mathcal{A}^+$  with least upper bound  $B$ , then  $\{A^{1/2}B_nA^{1/2} \mid n \in D\}$  is a monotonically increasing net in the trace class with least upper bound  $A^{1/2}BA^{1/2}$  in the trace class. This means that  $\text{lub}_n \phi(B_n) =$

$\text{lub}_n \text{Tr} (A^{1/2} B_n A^{1/2}) = \text{Tr} (A^{1/2} B A^{1/2}) = \phi(B)$ . This shows that  $\phi$  is ultraweakly continuous. Q.E.D.

We now extend the trace  $\text{Tr}$  to  $\mathcal{A}^+$ . Let  $\{P_j \mid j \in J\}$  be a set of mutually orthogonal central projections such that for each  $j$   $\mathcal{Z}_j = \mathcal{Z}P_j$  is  $\sigma$ -finite. If  $H$  is the Hilbert space of  $\mathcal{A}$ , let  $x_j$  be a separating vector in  $P_j H$  for  $\mathcal{Z}_j$ . Let  $Z_j$  be the spectrum of  $\mathcal{Z}_j$ . This is identified with the compact set  $\{\zeta \in Z \mid P_j^\wedge(\zeta) = 1\}$ . Let  $\nu_j$  be the measure on  $Z_j$  such that  $w_j(A) = (Ax_j, x_j) = \int_{Z_j} A^\wedge(\zeta) d\nu_j(\zeta)$  for every  $A \in \mathcal{Z}_j$ . Let  $X = \bigcup \{Z_j \mid j \in J\}$  with the topology induced by the union. Let  $\nu$  be the measure on  $X$  induced by each  $\nu_j$  acting on  $Z_j$ . Then  $\mathcal{Z}$  is isomorphic isometric to the algebra  $L_c^\infty(X, \nu)$  of all essentially bounded complex-valued functions on  $X$ .

For each  $j$  the function  $w_j \cdot \text{Tr} (A)$  on  $\mathcal{P}P_j$  is normal in the sense of Theorem 4.11. There is a unique faithful normal semifinite trace  $\phi_j$  on  $\mathcal{A}^+P_j$  such that  $\phi_j(A) = w_j \cdot \text{Tr} (A)$  for every  $A \in \mathcal{P}P_j$ . If  $F(J)$  denotes the collection of all finite subsets of  $J$ , the function of  $\mathcal{A}^+$  defined by

$$\phi(A) = \sup \left\{ \sum (\phi_j(A) \mid j \in \pi) \mid \pi \in F(J) \right\}$$

defines a faithful normal semifinite trace on  $\mathcal{A}^+$ . Let  $\mathcal{Z}^{*+}$  be the set of all positive (finite or infinite)  $\nu$ -measurable functions on  $X$ . Define the trace  $w$  on  $\mathcal{Z}^{*+}$  by  $w(f) = \sup \{\text{outer integral of } fg_Y \mid g_Y \text{ characteristic function of a compact set } Y\}$ . There is a faithful normal  $\#$ -map  $\Phi$  of  $\mathcal{A}^+$  into  $\mathcal{Z}^{*+}$  such that  $w \cdot \Phi(A) = \phi(A)$  for all  $A \in \mathcal{A}^+$ . If  $A \in \mathcal{P}P_j$ , then  $w_j(\text{Tr} (AP_j)) = \phi(A) = w\Phi(A) = w_j(\Phi(AP_j))$ . However, if  $B$  is any element in  $\mathcal{Z}_j^+$  then  $w_j(B \text{Tr} (A)) = w_j(B\Phi(A))$ ; so  $\text{Tr} (A) = \Phi(A)$ .

For any  $A \in \mathcal{P}$ ,  $\text{Tr} (A)P_j = \Phi(A)P_j$  for all  $j$ . Therefore,  $\Phi(A) \in \mathcal{Z}$  and  $\text{Tr} (A) = \Phi(A)$ . [2, III, §1, (Problem 11) and §4; I, §6 (Proposition 9)].

Let  $\phi$  be an ultraweakly continuous  $\mathcal{Z}$ -linear function of  $\mathcal{A}$  into  $\mathcal{Z}$ . The functions  $\phi_1(A) = 2^{-1}(\phi(A) + \phi(A^*)^*)$  and  $\phi_2(A) = (2i)^{-1}(\phi(A) - \phi(A^*)^*)$  are  $\mathcal{Z}$ -linear ultraweakly continuous functions such that  $\phi_1(A^*) = \phi_1(A)^*$  and  $\phi_2(A^*) = \phi_2(A)^*$ . We say that a  $\mathcal{Z}$ -linear function  $\phi$  on  $\mathcal{A}$  is hermitian if  $\phi(A^*) = \phi(A)^*$  for each  $A$  in  $\mathcal{A}$ . Thus, every ultraweakly continuous  $\mathcal{Z}$ -linear function may be written as a linear combination of two hermitian ultraweakly continuous  $\mathcal{Z}$ -linear functions.

The next theorem shows that each ultraweakly continuous hermitian  $\mathcal{Z}$ -linear function may be written as the difference of normal  $\mathcal{Z}$ -linear functions.

**THEOREM 4.12.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  on the Hilbert space  $H$ . Let  $\phi$  be a hermitian ultraweakly continuous  $\mathcal{Z}$ -linear function of  $\mathcal{A}$  into  $\mathcal{Z}$ . There is a projection  $E$  in  $\mathcal{A}$  such that  $\phi(AE) = \phi(EA)$  for every  $A$  in  $\mathcal{A}$  and such that the  $\mathcal{Z}$ -linear functions  $\phi_1(A) = \phi(AE)$  and  $\phi_2(A) = -\phi(A(1-E))$  of  $\mathcal{A}$  are normal.*

**Proof.** Let  $x$  be a nonzero vector in  $H$ . The linear functional  $\rho(A) = (\phi(A)x, x)$  on  $\mathcal{A}$  is a nonzero ultraweakly continuous hermitian linear functional. There is a projection  $F$  in  $\mathcal{A}$  such that  $\rho(FA) = \rho(AF)$  for every  $A$  in  $\mathcal{A}$  and such that  $A \rightarrow \rho(AF)$

and  $A \rightarrow -\rho(A(1-F))$  are normal functionals on  $\mathcal{A}$ . For every  $B$  in  $\mathcal{Z}$  and  $A$  in  $\mathcal{A}^+$  we have that

$$0 \leq (\phi(afb^*b)x, x) = (\phi(af)bx, bx) = (\phi(fa)bx, bx),$$

and

$$0 \leq -(\phi(A(1-F))b^*bx, x) = -(\phi(A(1-F))bx, bx).$$

Let  $G'$  be nonzero projection in the commutator  $\mathcal{Z}'$  of  $\mathcal{Z}$  that corresponds to the subspace closure  $\{Bx \mid B \in \mathcal{Z}\}$ . Since  $0 \leq (\phi(af)y, y) = (\phi(fa)y, y)$  and  $0 \leq -(\phi(A(1-F))y, y)$  for every  $A$  in  $\mathcal{A}^+$  and  $y$  in closure  $\{Bx \mid B \in \mathcal{Z}\}$ , we have that  $\phi(af)G' = \phi(fa)G'$  and  $-\phi(A(1-F))G'$  are positive central elements in  $G'\mathcal{Z}'G'$  for every  $A$  in  $\mathcal{A}^+$ . Let  $G$  be the projection in  $\mathcal{Z}$  corresponding to closure  $\{B'x \mid B' \in \mathcal{Z}'\}$ . The function  $A \rightarrow AG'$  of  $\mathcal{Z}G$  onto  $\mathcal{Z}G'$  is an isomorphism. So  $\phi(af)G = \phi(fa)G$  and  $-\phi(A(1-F))G$  are positive operators for each  $A$  in  $\mathcal{A}^+$ .

Now let  $\{G_n\}$  be a maximal net of nonzero orthogonal central projections such that for each  $n$  there is a projection  $E_n$  majorized by  $G_n$  such that  $\phi(AE_n) = \phi(E_nA)$  and  $-\phi(A(G_n - E_n))$  are positive for each  $A$  in  $\mathcal{A}^+$ . Let  $G = \sum G_n$  and  $E = \sum E_n$ . Then  $\phi(AE) = \phi(EA)$  and  $-\phi(A(G - E))$  are positive for each  $A$  in  $\mathcal{A}^+$  due to the ultraweak continuity of  $\phi$ . From the preceding paragraph we see that the maximality of  $\{G_n\}$  implies that  $G = 1$ . Q.E.D.

We now identify the elements of trace class with the ultraweakly continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{A}$  into  $\mathcal{Z}$ .

**THEOREM 4.13.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra and let  $\mathcal{Z}$  be the center of  $\mathcal{A}$ . Let  $\mathcal{P}$  be the set of positive elements of the trace class  $\mathcal{T}$  of  $\mathcal{A}$  and let  $\text{Tr}$  be the trace function. If  $A \in \mathcal{T}$ , let  $\phi_A$  be the function on  $\mathcal{A}$  defined by  $\phi_A(B) = \text{Tr}(BA)$ . The function  $A \rightarrow \phi_A$  is an isometric isomorphism of  $\mathcal{T}$  onto the set of all ultraweakly continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{A}$  into  $\mathcal{Z}$ . The function  $A \rightarrow \phi_A$  takes  $\mathcal{P}$  onto the set of all normal  $\mathcal{Z}$ -linear functions.*

**Proof.** Let  $\phi$  be an ultraweakly continuous  $\mathcal{Z}$ -linear function of  $\mathcal{A}$  into  $\mathcal{Z}$ . By Theorem 4.12 and the remarks preceding it the function  $\phi$  may be written as a linear combination of four normal  $\mathcal{Z}$ -linear functions  $\phi_j$  ( $1 \leq j \leq 4$ ). There is a constant  $K$  such that

$$\|\phi_j(B)\phi_j(B)^*\| \leq K\|\phi_j(B^*B)\| \leq K\|\phi_j(1)\| \|B\|^2$$

for every  $B$  in  $\mathcal{A}$ . Therefore, the functions  $\phi_j$  restricted to  $I_a$  are uniformly continuous  $\mathcal{Z}$ -linear functionals. There are unique elements  $A_j$  in  $\mathcal{T}$  such that  $\phi_j(B) = \text{Tr}(BA_j)$  for each  $B$  in  $I_a$ . For every abelian projection  $E$  in  $\mathcal{A}$  we have that  $0 \leq \phi_j(E) = \tau_E(A_j)$ . This shows that each  $A_j$  is an element of  $\mathcal{P}$ .

Finally, the functions  $\theta_j(B) = \text{Tr}(BA_j)$  are normal on  $\mathcal{A}$ . However, the ideal  $I_a$  is ultraweakly dense in  $\mathcal{A}$ . Since  $\theta_j$  is equal to  $\phi_j$  on a dense subset of  $\mathcal{A}$ , we have that  $\theta_j$  is equal to  $\phi_j$  for each  $j$ . Thus the function  $A \rightarrow \phi_A$  of  $\mathcal{T}$  maps onto the set of ultraweakly continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{A}$  into  $\mathcal{Z}$  and maps  $\mathcal{P}$  onto the

set of normal  $\mathcal{Z}$ -linear functions. This function is certainly a monomorphism. Furthermore, by Theorem 4.9 we have  $\|\text{Tr}(|A|)\| = \text{lub} \{\|\phi_A(B)\| \mid \|B\| \leq 1, B \in I_a\}$ . By the Kaplansky density theorem [6] the unit sphere of  $I_a$  is strongly dense in the unit sphere of  $\mathcal{A}$ . Let  $\varepsilon > 0$  be given; then there is an operator  $B$  in the unit sphere of  $\mathcal{A}$  and unit vectors  $x$  and  $y$  in the Hilbert space  $H$  of  $\mathcal{A}$  such that  $\|\phi_A\| \leq |(\phi_A(B)x, y)| + \varepsilon$ . Therefore, there is an element  $C$  in the unit sphere of  $I_a$  such that  $\|\phi_A\| \leq |(\phi_A(C)x, y)| + 2\varepsilon$ . So  $\|\phi_A\| \leq |(\phi_A(C)x, y)| + 2\varepsilon \leq \|\text{Tr}(|A|)\| + 2\varepsilon$ . Because  $\varepsilon > 0$  is arbitrary, we have that  $\|\phi_A\| \leq \|\text{Tr}(|A|)\|$ . Finally, we obtain that  $\|\phi_A\| = \|\text{Tr}(|A|)\|$ . Therefore, the function  $A \rightarrow \phi_A$  is an isometric isomorphism of  $\mathcal{T}$  onto the space of all ultraweakly continuous  $\mathcal{Z}$ -linear functions of  $\mathcal{A}$  into  $\mathcal{Z}$ . Q.E.D.

Let  $\mathcal{A}$  be a semifinite von Neumann algebra with center  $\mathcal{Z}$ . Let  $X$  be a locally compact topological space and let  $\nu$  be a positive measure on  $X$  so that the set of all essentially bounded measurable functions  $L_c^\infty(X, \nu)$  on  $X$  is isometrically isomorphic to  $\mathcal{Z}$ . Let  $\Phi$  be a normal semifinite faithful  $\#$ -map of  $\mathcal{A}^+$  into the set of all positive (finite or infinite valued) measurable functions on  $X$ . Let  $\mathcal{N} = \{A \in \mathcal{A} \mid \Phi(A^*A) \in \mathcal{Z}\}$  and  $\mathcal{M} = \mathcal{N}^2$ . For  $A$  and  $B \in \mathcal{N}$ , let  $(A, B) = \Phi(B^*A)$  and let  $\|A\|_1 = \|\Phi(A^*A)^{1/2}\|$ . The function  $A \rightarrow \|A\|_1$  on  $\mathcal{N}$  is a norm. The following theorem gives part of our reason for not studying the subject of the preceding section in a more general context.

**THEOREM 4.14.** *Let  $\Phi$  be a normal semifinite faithful  $\#$ -map on the von Neumann algebra  $\mathcal{A}$  with center  $\mathcal{Z}$ ; then the ideal  $\mathcal{N} = \{A \in \mathcal{A} \mid \Phi(A^*A) \in \mathcal{Z}\}$  is complete under the norm  $\|A\|_1 = \|\Phi(A^*A)^{1/2}\|$  if and only if there is an  $\alpha > 0$  such that  $\|A\| \leq \alpha\|A\|_1$  for all  $A \in \mathcal{N}$ . In particular  $\mathcal{A}$  is of Type I.*

**Proof.** Let  $\mathcal{N}$  be complete under the norm  $\|A\|_1$ . There is no loss of generality in assuming either that  $\mathcal{A}$  is of Type I or that  $\mathcal{A}$  is of Type II. Suppose first that  $\mathcal{A}$  is of Type I and there is no  $\alpha > 0$  such that  $\|A\| \leq \alpha\|A\|_1$  for all  $A \in \mathcal{N}$ . There is a sequence  $\{A_n\}$  of elements of  $\mathcal{N}$  such that  $\|A_n\| = 1$  and  $\|A_n\|_1 < 2^{-(2n+1)}$ . We may assume that  $A_n \in \mathcal{A}^+$  for each  $n$ . There is for each  $n$  a nonzero abelian projection  $E_n$  such that  $A_n^2 \geq 2^{-1}E_n$ . Thus

$$\|E_n\|_1 = \|\Phi(E_n)^{1/2}\| \leq 2^{1/2}\|\Phi(A_n^2)^{1/2}\| \leq 2^{-2n}.$$

Let  $P_n$  be the central support of  $E_n$  and let  $X_n$  be the set in the spectrum  $Z$  of  $\mathcal{Z}$  to which  $P_n$  corresponds. We consider two cases: (1) there is a subsequence  $\{X_{n_j}\}$  of  $\{X_n\}$  such that  $\bigcap \{X_{n_j} \mid j=1, 2, \dots\} \neq \emptyset$ ; (2) for every subsequence  $\{X_{n_j}\}$  of  $\{X_n\}$ ,  $\bigcap \{X_{n_j} \mid j=1, 2, \dots\} = \emptyset$ . In case (1) let  $\bigcap \{X_{n_j} \mid j=1, 2, \dots\} \neq \emptyset$ . Let  $Q_j = P_{n_1}P_{n_2} \cdots P_{n_j}$  and let  $F_j = Q_jE_{n_j}$ . Then  $\{F_n\}$  is a sequence of nonzero abelian projections such that  $\|F_n\|_1 \leq 2^{-2n}$  and  $F_{n+1} < F_n$ . We may assume therefore that  $F_{n+1} < F_n$  ( $n=1, 2, \dots$ ) by choosing inductively appropriate projections equivalent to each of the original  $F_n$  and by passing to a subsequence if necessary. Let  $G_n = F_n - F_{n+1}$  ( $n=1, 2, \dots$ ). Then  $\{G_n\}$  is a sequence of nonzero disjoint abelian

projections such that  $\|G_n\|_1 \leq 2^{-2^n}$ . Now consider case (2). There is a sequence  $\{\pi_j\}$  of nonempty finite disjoint subsets of natural numbers with the following properties: (1) if  $i < j$  and  $k \in \pi_i, l \in \pi_j$  then  $k < l$ ; (2)  $Q_j = \times \{P_k \mid k \in \pi_j\} \neq 0$  for each  $j$ ; and (3) if  $k > j$  then  $P_l$  is orthogonal to  $Q_j$  for each  $l \in \pi_k$ . Let  $m(j)$  be the largest integer in  $\pi_j$  and let  $G_j = Q_j E_{m(j)}$ . Then  $\{G_n\}$  is a sequence of nonzero disjoint abelian projections such that  $\|G_n\|_1 \leq 2^{-2^n}$ .

Now for either case let  $B_n = \sum \{2^k G_k \mid 1 \leq k \leq n\}$  for  $n = 1, 2, \dots$ . Since

$$\Phi((B_m - B_n)^*(B_m - B_n))^{1/2} = \left( \sum \{2^{2j} \Phi(G_j) \mid m+1 \leq j \leq n\} \right)^{1/2}$$

for  $m < n$ , we have

$$\|B_m - B_n\|_1 \leq \left( \sum \{2^{2j} \Phi(G_j) \mid m+1 \leq j \leq n\} \right)^{1/2} \leq 2^{-m}.$$

Thus, the sequence  $\{B_n\}$  is Cauchy in  $\mathcal{N}$ . By assumption there is a  $B$  in  $\mathcal{N}$  such that  $\lim_n \|B_n - B\|_1 = 0$ . However, the sequence  $\{B_n \sum (G_k \mid 1 \leq k \leq m)\}_n$  which is eventually constant converges to  $\sum (G_k \mid 1 \leq k \leq m)B$  in  $\mathcal{N}$ . So  $\sum (G_k \mid 1 \leq k \leq m)B = \sum (2^k G_k \mid 1 \leq k \leq m)$  and hence  $\|B\| \geq 2^m$ . Because  $m$  is arbitrary this is impossible. We have proved that if  $\mathcal{A}$  is of Type I, and  $\mathcal{N}$  is complete there is an  $\alpha > 0$  such that  $\|A\| \leq \alpha \|A\|_1$  for every  $A$  in  $\mathcal{N}$ .

Let  $\mathcal{A}$  be a Type II algebra and let  $E$  be a nonzero projection in  $\mathcal{N}$ . Let  $F_{01} = E$  and let  $\{F_{nj} \mid n = 0, 1, \dots; 1 \leq j \leq 16\}$  be projections in  $\mathcal{A}$  such that  $\{F_{nj} \mid 1 \leq j \leq 16\}$  is a set of orthogonal equivalent projections of sum  $F_{n-11}$ , for each  $n = 1, 2, \dots$ . Then  $\Phi(F_{n-11}) = \sum_j \Phi(F_{nj}) = 16\Phi(F_{n1})$ . Let  $B_n = \sum \{2^j F_{j2} \mid 1 \leq j \leq n\}$ . We have that  $\{B_n\}$  is a Cauchy sequence in  $\mathcal{N}$ . Indeed

$$\begin{aligned} \|B_m - B_n\|_1 &= \left\| \left[ \sum \{2^{2j} \Phi(F_{j2}) \mid m+1 \leq j \leq n\} \right]^{1/2} \right\| \\ &= \left\| \sum \{2^{2j} 2^{-4j} \Phi(E) \mid m+1 \leq j \leq n\} \right\|^{1/2} \\ &\leq 2^{-m} \|\Phi(E)\|^{1/2} \end{aligned}$$

whenever  $m < n$ . Now there is a  $B \in \mathcal{N}$  such that  $\lim_n B_n = B$  in  $\mathcal{N}$ . However, the sequence  $\{B_n \sum \{F_{j2} \mid 1 \leq j \leq m\}\}_n$  which is eventually constant converges to  $\sum \{F_{j2} \mid 1 \leq j \leq m\}B$ . So  $\sum \{2^j F_{j2} \mid 1 \leq j \leq m\} = \sum \{F_{j2} \mid 1 \leq j \leq m\}B$ . Thus  $\|B\| \geq 2^m$ . This is impossible. Hence  $\mathcal{A}$  cannot be of Type II.

Conversely, suppose there is an  $\alpha > 0$  such that  $\|A\| \leq \alpha \|A\|_1$  for every  $A \in \mathcal{N}$ . Let  $\{A_n\}$  be a Cauchy sequence in  $\mathcal{N}$ . We show that  $\{A_n\}$  converges to an element  $A$  in  $\mathcal{A}$  using the method of Theorem 4.6. Because  $\{A_n\}$  is Cauchy, the sequence  $\{A_n\}$  converges uniformly to  $A \in \mathcal{A}$ . Let  $S$  be a set of orthogonal abelian projections in  $\mathcal{N}$  with least upper bound 1. Let  $\pi$  be a finite subset of  $S$ . Then the relation

$$\begin{aligned} &\left[ \sum \{ \tau_F((A - A_n)^*(A - A_n)) \Phi(F) \mid F \in \pi \} \right]^{1/2} \\ (1) \quad &\leq \left[ \sum \{ \tau_F((A - A_m)^*(A - A_m)) \Phi(F) \mid F \in \pi \} \right]^{1/2} \\ &\quad + \left[ \sum \{ \tau_F((A_m - A_n)^*(A_m - A_n)) \Phi(F) \mid F \in \pi \} \right]^{1/2}, \end{aligned}$$



which is obtained in a manner entirely similar to that in which the corresponding formula of Theorem 4.5 is obtained, gives

$$\|(A - A_n)F_\pi\|_1 \leq \|(A - A_m)F_\pi\|_1 + \|(A_m - A_n)F_\pi\|_1$$

where  $F_\pi = \sum \{F \mid F \in \pi\}$ . So there is a constant  $M$  such that  $\|(A - A_n)F_\pi\|_1 \leq M$  for every finite subset  $\pi$  of  $S$ . Therefore,  $A - A_n \in \mathcal{N}$  and consequently  $A \in \mathcal{N}$ . Furthermore, the relation (1) shows that  $\lim_n \|A - A_n\|_1 = 0$ . Q.E.D.

We now are able to show that the theory obtained by using a normal semifinite faithful  $\#$ -map with values in the center is different from the theory obtained by using a trace with values in the scalar field. Let  $\phi$  be a normal semifinite faithful trace (with scalar values) on a Type I von Neumann algebra  $\mathcal{A}$ . Let  $\mathcal{N} = \{A \in \mathcal{A} \mid \phi(A^*A) < +\infty\}$  and let  $\mathcal{M} = \mathcal{N}^2$ . Define the norm on  $\mathcal{N}$  by  $\|A\|_1 = \phi(A^*A)^{1/2}$ . If  $\mathcal{N}$  is complete with this norm, a simple reworking of the proof of Theorem 4.14 shows that there is an  $\alpha > 0$  such that  $\|A\| \leq \alpha \|A\|_1$  for all  $A \in \mathcal{N}$ . By a theorem of Ogasawara and Yoshinaga [11, Theorem 5]  $\mathcal{A}$  contains minimal abelian projections. (In fact  $\mathcal{A}$  is the product of factors.) This is not the situation if  $\phi$  is a normal semifinite faithful  $\#$ -map with values in the center of  $\mathcal{A}$ .

**THEOREM 4.15.** *Let  $\mathcal{A}$  be a Type I von Neumann algebra with center  $\mathcal{Z}$  and let  $\Phi$  be a faithful normal semifinite  $\#$ -map on  $\mathcal{A}$ . Let  $\mathcal{N} = \{A \in \mathcal{A} \mid \Phi(A^*A) \in \mathcal{Z}\}$  and let  $\mathcal{M} = \mathcal{N}^2$ . If  $\mathcal{N}$  is complete with the norm  $\|A\|_1 = \|\Phi(A^*A)^{1/2}\|$ , the ideal  $\mathcal{N}$  is contained in  $\mathcal{S}$  and the ideal  $\mathcal{M}$  is contained in  $\mathcal{T}$ . There is a  $B_0 \in \mathcal{Z}^+$  such that  $A \rightarrow AB_0^{1/2}$  is an isometric isomorphism of  $\mathcal{S}$  onto  $\mathcal{N}$  and  $A \rightarrow AB_0$  is an isometric isomorphism of  $\mathcal{T}$  onto  $\mathcal{M}$  with the norm  $\|A\|_2 = \|\Phi(|A|)\|$ .*

**Proof.** Let  $\alpha > 0$  be a number such that  $\|A\| \leq \alpha \|A\|_1$  for every  $A \in \mathcal{N}$ . Let  $F$  be a projection in  $\mathcal{N}$  and let  $P$  be the central support of  $F$ . We have that  $F \in I_a$  and  $\Phi(F) \geq \alpha' P$ , where  $\alpha' = \alpha^{-2}$ . So there is a  $B$  in  $(\mathcal{Z}P)^+$  such that  $\|B\| \leq \alpha'$  and  $B\Phi(F) = P$ . Let  $\{F_i\}$  be a set of mutually orthogonal abelian projections in  $\mathcal{N}$  whose least upper bound  $F$  is a maximal abelian projection. Let  $P_i$  be the central support of  $F_i$  and let  $B_i \in (\mathcal{Z}P_i)^+$  have the property  $B_i\Phi(F_i) = P_i$  and  $\|B_i\| \leq \alpha'$ . There is a  $B_0 \in \mathcal{Z}^+$  such that  $B_0P_i = P_i$  for each  $i$ . Because  $B_0\Phi(F) = \text{Tr}(F)$ ,  $B_0\Phi(A) = \text{Tr}(A)$  for every  $A \in \mathcal{A}^+$  [2, III, §4, Theorem 2]. If  $E$  is any abelian projection in  $\mathcal{N}$ , then  $B_0\Phi(E) = P$  where  $P$  is the central support of  $E$ . Thus,  $B_0^{1/2}\Phi(E)^{1/2} = P$ . Thus  $A \in \mathcal{S}$  implies  $AB_0^{1/2} \in \mathcal{N}$  and  $\Phi((AB_0^{1/2})^*(AB_0^{1/2}))^{1/2} = \text{Tr}(A^*A)^{1/2}$ . Also if  $A \in \mathcal{T}$  then  $AB_0 \in \mathcal{M}$  and  $\Phi(|AB_0|) = \text{Tr}(|A|)$ . So  $A \rightarrow AB_0^{1/2}$  and  $A \rightarrow AB_0$  are isometric isomorphic functions of  $\mathcal{S}$  into  $\mathcal{N}$  and  $\mathcal{T}$  into  $\mathcal{M}$  respectively.

Let  $A \in \mathcal{N}^+$ . Since every projection of  $\mathcal{N}$  is an element of  $I_a$ , the element  $A$  is a member of  $I_a$ . If  $\sum A_j E_j$  is a spectral resolution for  $A$ , then  $A^*A = A^2 = \sum A_j^2 E_j$  and  $\Phi(A^*A) = \sum A_j^2 \Phi(E_j)$ . Now  $\sum A_j \Phi(E_j)^{1/2} E_j$  is an element of  $\mathcal{S}$ . Indeed, if  $B_n = \sum \{A_j \Phi(E_j)^{1/2} E_j \mid 1 \leq j \leq n\}$ , the sequence  $\{B_n\}$  converges uniformly to an element  $B$  in  $I_a^+$  which has a spectral resolution  $\sum A_j \Phi(E_j) E_j$ . The relation  $E_1$

$\succ E_2 \cdots$  implies  $\Phi(E_1)^{1/2} \geq \Phi(E_2)^{1/2} \geq \cdots$  and the relation  $\alpha \|G\|_1 \geq \|G\|$  for every projection  $G \in \mathcal{N}$  implies that the product  $A_j \Phi(E_j)^{1/2} E_j$  has property (iv) of Theorem 2.2. Also  $\lim A_j \Phi(E_j)^{1/2} = 0$ . We have that  $BB_0^{1/2} = \sum A_j B_0^{1/2} \Phi(E_j)^{1/2} E_j = \sum A_j E_j$ . Therefore, the function  $A \rightarrow AB_0^{1/2}$  maps  $\mathcal{S}$  onto  $\mathcal{N}$ .

If  $A \in \mathcal{M}^+$ , then  $A$  is an element of  $I_a$ . Let  $\sum A_j E_j$  be a spectral resolution of  $A$ ; we have  $\Phi(|A|) = \Phi(A) = \sum A_j \Phi(E_j)$ . In the same manner as the preceding paragraph  $\sum \{A_j \Phi(E_j) E_j \mid 1 \leq j \leq n\}$  converges uniformly to an element  $B$  in  $I_a^+$  whose spectral resolution is  $\sum A_j \Phi(E_j) E_j$ . Then  $B \in \mathcal{T}$  and  $B_0 B = \sum A_j E_j$ . This shows that the function  $A \rightarrow AB_0$  of  $\mathcal{T}$  into  $\mathcal{M}$  is onto  $\mathcal{M}$ . Q.E.D.

COROLLARY. If  $\mathcal{M} = \mathcal{T}$ , then  $B_0$  is invertible.

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